CONNECTEDNESS OF THE BALMER SPECTRUM OF THE RIGHT BOUNDED DERIVED CATEGORY

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Abstract. By virtue of Balmer's celebrated theorem, the classification of thick tensor ideals of a tensor triangulated category $\mathcal{T}$ is equivalent to the topological structure of its Balmer spectrum $\mathtt{Spc}\,\mathcal{T}$. Motivated by this theorem, we discuss connectedness, irreducibility, and Noetherianity of the Balmer spectrum of a right bounded derived category of finitely generated modules over a commutative ring.

1. Introduction

Tensor triangulated geometry is a theory introduced by Balmer [1] to study tensor triangulated categories by algebro-geometric methods. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be an essentially small tensor triangulated category (i.e., a triangulated category $\mathcal{T}$ equipped with a symmetric monoidal tensor product $\otimes$ which is compatible with the triangulated structure). Then Balmer defined a topological space $\mathtt{Spc}\,\mathcal{T}$ which we call the Balmer spectrum of $\mathcal{T}$. A celebrated theorem due to Balmer [1] states that the radical thick tensor ideals of $\mathcal{T}$ are classified using the geometry of $\mathtt{Spc}\,\mathcal{T}$:

Theorem 1.1 (Balmer)**.** *There is an order-preserving one-to-one correspondence*

$$\{\textit{radical thick tensor ideals of } \mathcal{T}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{Thomason subsets of } \mathtt{Spc}\,\mathcal{T}\},$$

where f and g are given by $f(\mathcal{X}) := \mathsf{BSupp}\,\mathcal{X} := \bigcup_{X\in\mathcal{X}} \mathsf{BSupp}\,X$ and $g(W) := \mathsf{BSupp}^{-1}(W) := \{X \in \mathcal{T} \mid \mathsf{BSupp}\,X \subseteq W\}$, respectively.

From this result, if we want to classify the radical thick tensor ideals of a given tensor triangulated category $\mathcal{T}$, we have only to understand the topological space $\mathtt{Spc}\,\mathcal{T}$. Therefore, it is crucial to discuss topological properties of the Balmer spectrum.

In this paper, we consider the right bounded derived category $\mathsf{D}^-(\mathsf{mod}\,R)$ of a commutative Noetherian ring R. This triangulated category is a tensor triangulated category with respect to derived tensor product, and we can consider its Balmer spectrum $\mathtt{Spc}\,\mathsf{D}^-(\mathsf{mod}\,R)$. The main results of this paper are the following two theorems:

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Theorem 1.2 (Theorem 3.1)**.** *If the Balmer spectrum* $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}\,R)$ *is a Noetherian topological space, then the Zariski spectrum* $\mathsf{Spec}\,R$ *is a finite set.*

Theorem 1.3 (Corollary 3.12)**.** *The Balmer spectrum* $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}\,R)$ *is connected if and only if the Zariski spectrum* $\mathsf{Spec}\,R$ *is so.*

Moreover, by using the latter theorem, we give a variant of a well-known result of Carlson [5] in representation theory.

This paper is organized as follows. In Section 2, we recall some basic materials from tensor triangulated geometry and point-set topology. In Section 3, we prove our main theorems and give an application. In Section 4, we discuss realizing a clopen subset of $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}\,R)$ as a Balmer spectrum.

2. Preliminaries

Throughout this paper, let R be a commutative Noetherian ring. For an ideal I of R, we denote by $\mathsf{V}(I)$ the ideals of R containing I. We note that $\mathsf{V}(I)$ is a closed subset of the Zariski spectrum $\mathsf{Spec}\,R$ and $\mathsf{V}(\mathfrak{p})$ is the closure $\overline{\{\mathfrak{p}\}}$ of $\mathfrak{p}$ in $\mathsf{Spec}\,R$. Denote by $\mathsf{D}^-(R)$ (resp. $\mathsf{D}^\mathsf{b}(R)$) the derived category of complexes M of finitely generated R-modules with $\mathsf{H}^i(M) = 0$ for all $i \gg 0$ (resp. $|i| \gg 0$). Then $\mathsf{D}^-(R)$ is an essentially small tensor triangulated category via derived tensor product $\otimes_R^{\mathbf{L}}$ with unit R.

First we will recall the definitions of a thick tensor ideal, a radical thick tensor ideal, and a prime thick tensor ideal.

Definition 2.1. Let $\mathcal{T}$ be an essentially small tensor triangulated category.

(1) A subcategory $\mathcal{X}$ of $\mathcal{T}$ is called a *thick tensor ideal* of $\mathcal{T}$ if it is a thick subcategory of $\mathcal{T}$ and for any $M \in \mathcal{T}$ and $N \in \mathcal{X}$, the tensor product $M \otimes N$ belongs to $\mathcal{X}$.
(2) For a thick tensor ideal $\mathcal{X}$ of $\mathcal{T}$, we denote by $\sqrt{\mathcal{X}}$ the *radical* of $\mathcal{X}$, that is, the subcategory of $\mathcal{T}$ consisting of objects M such that the n-fold tensor product $M \otimes M \otimes \cdots \otimes M$ belongs to $\mathcal{X}$ for some integer $n \geq 1$.
(3) A thick tensor ideal $\mathcal{X}$ of $\mathcal{T}$ is called *radical* if $\sqrt{\mathcal{X}} = \mathcal{X}$.
(4) A proper thick tensor ideal $\mathcal{P}$ of $\mathcal{T}$ is called *prime* if $M \otimes N \in \mathcal{P}$ implies either $M \in \mathcal{P}$ or $N \in \mathcal{P}$. The set of prime thick tensor ideals of $\mathcal{T}$ is denoted by $\mathsf{Spc}\,\mathcal{T}$ and we call it the *Balmer spectrum* of $\mathcal{T}$.

For a thick tensor ideal $\mathcal{X}$, its radical $\sqrt{\mathcal{X}}$ is a thick tensor ideal. Indeed, by [1, Lemma 4.2], it is equal to the intersection of all prime thick tensor ideals containing $\mathcal{X}$.

Example 2.2. For a complex $M \in \mathsf{D}^-(R)$, define the *support* of M by

$$\begin{aligned}\mathsf{Supp}\,M &:= \{\mathfrak{p} \in \mathsf{Spec}\,R \mid M_\mathfrak{p} \ncong 0 \text{ in } \mathsf{D}^-(R_\mathfrak{p})\} \\ &= \bigcup_{n\in\mathbb{Z}} \mathsf{Supp}\,\mathsf{H}^n(M).\end{aligned}$$

Moreover, for a class $\mathcal{X}$ of objects of $\mathsf{D}^-(R)$, denote by $\mathsf{Supp}\,\mathcal{X} := \bigcup_{M\in\mathcal{X}} \mathsf{Supp}\,M$. Then, for a subset W of $\mathsf{Spec}\,R$,

$$\mathsf{Supp}^{-1}\,W := \{M \in \mathsf{D}^-(R) \mid \mathsf{Supp}\,M \subseteq W\}$$

is a thick tensor ideal of $\mathsf{D}^-(R)$. Furthermore, if we take $W := \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$ for a fixed $\mathfrak{p}$, then

$$\mathcal{S}(\mathfrak{p}) := \operatorname{Supp}^{-1} W = \{M \in \mathsf{D}^-(R) \mid M_\mathfrak{p} \cong 0 \text{ in } \mathsf{D}^-(R_\mathfrak{p})\}$$

is a prime thick tensor ideal of $\mathsf{D}^-(R)$ by [7, Proposition 3.4].

Balmer [1] defined a topology on $\operatorname{Spc} \mathcal{T}$ as follows.

Definition 2.3. (1) For an object $M \in \mathcal{T}$, the *Balmer support* of M, denoted by $\operatorname{BSupp} M$, is defined as the set of prime thick tensor ideals not containing M. Set $\mathtt{U}(M) := \operatorname{Spc} \mathcal{T} \setminus \operatorname{BSupp} M$.
(2) Define a topology on $\operatorname{Spc} \mathcal{T}$ whose open basis is $\{\mathtt{U}(X) \mid X \in \mathcal{T}\}$.

We always consider this topology on the Balmer spectrum.
Next, let us recall some notions from point-set topology for later use.

Definition 2.4. Let X be a topological space.

(1) We say that a subspace of X is a *clopen subset* if it is closed and open in X.
(2) A subspace W of X is said to be *specialization closed* if for any $x \in W$, $\overline{\{x\}} \subseteq W$ holds.
(3) A subspace W of X is said to be *generalization closed* if for any $x \in W$ and $y \in X$, $x \in \overline{\{y\}}$ implies $y \in W$.
(4) We say that X is *connected* if it contains no non-trivial clopen subset. For a subspace Y of X, we say that Y is a *connected subspace* of X if it is a connected space by induced topology. Moreover, a *connected component* of X is a maximal connected subspace of X.
(5) We say that X is *irreducible* if it is non-empty and not the union of two proper closed subspaces. For a subspace Y of X, we say that Y is an *irreducible subspace* of X if it is an irreducible space by induced topology. Moreover, an *irreducible component* of X is a maximal irreducible subspace of X, which is automatically closed since the closure of irreducible subspace is also irreducible.
(6) We say that X is *Noetherian* if every descending chain of closed subspaces stabilizes.

Remark 2.5. (1) A subspace is generalization closed if and only if its complement is specialization closed. In particular, every open subset of X is generalization closed.
(2) Let $X \supseteq Y \supseteq Z$ be subspaces. If Y is specialization closed in X and Z is specialization closed in Y, then Z is specialization closed in X.
(3) Let W be a subspace of $\operatorname{Spec} R$. Then W is specialization closed (resp. generalization closed) in $\operatorname{Spec} R$ if and only if

$$\mathfrak{p} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \Longrightarrow \mathfrak{q} \in W$$
$$(\text{resp. } \mathfrak{q} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \Longrightarrow \mathfrak{p} \in W).$$

(4) [1, Proposition 2.9] Let $\mathcal{T}$ be an essentially small tensor triangulated category and W a subspace of $\operatorname{Spc} \mathcal{T}$. Then W is specialization closed (resp. generalization closed)

in $\mathsf{Spc}\,\mathcal{T}$ if and only if

$$\mathcal{P} \in W,\ \mathcal{P} \supseteq \mathcal{Q} \Longrightarrow \mathcal{Q} \in W$$
$$(\text{resp. } \mathcal{Q} \in W,\ \mathcal{P} \supseteq \mathcal{Q} \Longrightarrow \mathcal{P} \in W).$$

Lemma 2.6. *Let X be a topological space. Then every connected component of X is both specialization closed and generalization closed.*

Proof. Fix a connected component O of X. For $x \in O$, $\overline{\{x\}}$ is irreducible and in particular connected. Since $O \cap \overline{\{x\}}$ is non-empty, $O \cup \overline{\{x\}}$ is connected. Thus, $O \cup \overline{\{x\}}$ must be equal to O, and hence $\overline{\{x\}} \subseteq O$. This shows that O is specialization closed in X.

For $x \notin O$, assume that there exists $y \in \overline{\{x\}}$ with $y \in O$. Then $\overline{\{x\}} \cap O$ is non-empty as it contains y. Therefore, the same argument as above shows that $\overline{\{x\}} \subseteq O$. This gives a contradiction to $x \notin O$. Thus, $X \setminus O$ is specialization closed in X and hence O is generalization closed in X. ■

3. Main theorems

In this section, we discuss Noetherianity, connectedness, and irreducibility of the Balmer spectrum $\mathsf{Spc}\,\mathsf{D}^-(R)$.

3.1. **Noetherianity.** Besides, we show the following theorem which gives a sufficient condition for Noetherianity of the Balmer spectrum $\mathsf{Spc}\,\mathsf{D}^-(R)$.

Theorem 3.1. *If the Balmer spectrum $\mathsf{Spc}\,\mathsf{D}^-(R)$ is a Noetherian topological space, then $\mathsf{Spec}\,R$ is a finite set (i.e., semi-local ring with Krull dimension at most 1).*

Before proving this, we give the following easy lemma.

Lemma 3.2. *If $\mathsf{Spec}\,R$ has infinitely many prime ideals, then there is a countable antichain of prime ideals.*

Proof. If R has infinitely many maximal ideals, then we can take such a set as a countable set of pairwise non-equal maximal ideals.

Assume that R has only finitely many maximal ideals. Then R has finite Krull dimension. Since R has infinitely many prime ideals, there is a non-negative integer n such that the set $\{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathsf{ht}\,\mathfrak{p} = n\}$ has infinitely many elements. Thus, a countable subset of this set has the desired property. ■

For a complex $M \in \mathsf{D}^-(R)$, denote by $\langle M \rangle$ the smallest thick tensor ideal of $\mathsf{D}^-(R)$ containing M.

Proof of Theorem 3.1. Assume that $\mathsf{Spc}\,\mathsf{D}^-(R)$ is Noetherian. Then for any chain of the form $\mathsf{BSupp}\,M_1 \supseteq \mathsf{BSupp}\,M_2 \supseteq \mathsf{BSupp}\,M_3 \cdots$ with $M_i \in \mathsf{D}^-(R)$ stabilizes. Thus, by Theorem 1.1, every descending chain $\sqrt{\langle M_1 \rangle} \supseteq \sqrt{\langle M_2 \rangle} \supseteq \sqrt{\langle M_3 \rangle} \supseteq \cdots$ stabilizes.

Assume furthermore that R has infinitely many prime ideals. From the previous lemma, we can take a countable antichain $\{\mathfrak{p}_n\}_{n\geq 1}$ of prime ideals. Set $M_n := \bigoplus_{i\geq n} R/\mathfrak{p}_i[i]$ to be the complex

$$M_n := (\cdots \xrightarrow{0} R/\mathfrak{p}_{n+2} \xrightarrow{0} R/\mathfrak{p}_{n+1} \xrightarrow{0} R/\mathfrak{p}_n \to 0 \cdots).$$

Here, $R/\mathfrak{p}_i$ fit into the $(-i)$-th component. Then M_n belongs to $\mathsf{D}^-(R)$, and M_{n+1} is a direct summand of M_n for each integer $n \geq 1$. Therefore, we have a descending chain $\sqrt{\langle M_1\rangle} \supseteq \sqrt{\langle M_2\rangle} \supseteq \sqrt{\langle M_3\rangle} \supseteq \cdots$ of radical thick tensor ideals. From the above argument, we get an equality $\sqrt{\langle M_n\rangle} = \sqrt{\langle M_{n+1}\rangle}$ for some integer $n \geq 1$. Taking Supp, we obtain

$$\bigcup_{i\geq n} \mathsf{V}(\mathfrak{p}_i) = \mathsf{Supp}\sqrt{\langle M_n\rangle} = \mathsf{Supp}\sqrt{\langle M_{n+1}\rangle} = \bigcup_{i\geq n+1} \mathsf{V}(\mathfrak{p}_i).$$

Hence, there is an integer $m \geq n+1$ such that $\mathfrak{p}_m \subseteq \mathfrak{p}_n$. This gives a contradiction. ■

Remark 3.3. If R is Artinian, then by [7, Theorem 6.5], $\mathsf{Spc}\,\mathsf{D}^-(R)$ is homeomorphic to $\mathsf{Spec}\,R$. In particular, $\mathsf{Spc}\,\mathsf{D}^-(R)$ is a Noetherian topological space.

3.2. **Connectedness.** In this subsection, we mainly discuss connectedness of the Balmer spectrum $\mathsf{Spc}\,\mathsf{D}^-(R)$. We use the following pair of maps defined in [7] to compare two spectra:

$$\mathfrak{s} : \mathsf{Spc}\,\mathsf{D}^-(R) \rightleftarrows \mathsf{Spec}\,R : \mathcal{S}.$$

Here the map $\mathcal{S}$ was defined in Example 2.2 and $\mathfrak{s}(\mathcal{P})$ is the unique maximal element of the set of ideals I of R with $R/I \notin \mathcal{P}$, see [7, Proposition 3.7]. Let me list some basic properties of these maps in the following proposition.

Proposition 3.4. [7, Theorem 3.9, Corollary 3.10, Theorem 4.5]

(1) *Both maps $\mathfrak{s}$ and $\mathcal{S}$ are order-reversing.*
(2) *$\mathfrak{s}$ is continuous.*
(3) *$\mathfrak{s} \cdot \mathcal{S} = 1$. In particular, $\mathfrak{s}$ is surjective and $\mathcal{S}$ is injective.*
(4) *For a prime thick tensor ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$, one has*

$$\mathcal{S}\mathfrak{s}(\mathcal{P}) = \mathsf{Supp}^{-1}\,\mathsf{Supp}(\mathcal{P}) \supseteq \mathcal{P}.$$

(5) *For a prime thick tensor ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$, one has*

$$\mathsf{Supp}\,\mathcal{P} = \{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\mathcal{P})\}.$$

Remark 3.5. As it has been shown in [7, Theorem 4.7], $\mathcal{S}$ is not continuous in general.

The following theorem is the main result of this subsection.

Theorem 3.6. *Let $C \in \mathsf{D}^b(R)$ be a bounded complex.*

(1) *There is a one-to-one correspondence*

$$\left\{\textit{connected components of } \mathsf{BSupp}\,C\right\} \underset{\mathfrak{s}^{-1}}{\overset{\mathfrak{s}}{\rightleftarrows}} \left\{\textit{connected components of } \mathsf{Supp}\,C\right\}.$$

(2) *There is a one-to-one correspondence*

$$\left\{\textit{irreducible components of } \mathsf{BSupp}\, C\right\} \underset{\mathfrak{s}^{-1}}{\overset{\mathfrak{s}}{\rightleftarrows}} \left\{\textit{irreducible components of } \mathsf{Supp}\, C\right\}.$$

The proof of this theorem is divided into several lemmata.

Fix a bounded complex $C \in \mathsf{D}^{\mathsf{b}}(R)$. Then by [7, Proposition 2.9], for a thick tensor ideal $\mathcal{X}$, $C \in \mathcal{X}$ if and only if $\mathsf{Supp}\, C \subseteq \mathsf{Supp}\, \mathcal{X}$. By combining this with Proposition 3.4, $\mathcal{P} \in \mathsf{BSupp}\, C$ if and only if $\mathcal{S}\mathfrak{s}(\mathcal{P}) \in \mathsf{BSupp}\, C$ for a prime thick tensor ideal $\mathcal{P}$ of $\mathsf{D}^{-}(R)$.

Lemma 3.7. (1) $\mathfrak{s}(\mathsf{BSupp}\, C) = \mathsf{Supp}\, C$.
(2) $\mathcal{S}(\mathsf{Supp}\, C) \subseteq \mathsf{BSupp}\, C$.

Proof. (1) For a prime thick tensor ideal $\mathcal{P}$ in $\mathsf{BSupp}\, C$, we have the following equivalences:

$$\begin{aligned} \mathcal{P} \in \mathfrak{s}^{-1}(\mathsf{Supp}\, C) &\Leftrightarrow \mathfrak{s}(\mathcal{P}) \in \mathsf{Supp}\, C \\ &\Leftrightarrow \mathsf{Supp}\, C \not\subseteq \mathsf{Supp}\, \mathcal{P} = \{\mathfrak{p} \in \mathsf{Spec}\, R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\mathcal{P})\} \\ &\Leftrightarrow C \notin \mathcal{P} \\ &\Leftrightarrow \mathcal{P} \in \mathsf{BSupp}\, C. \end{aligned}$$

Here, the first and the last equivalences are clear. Since $\{\mathfrak{p} \in \mathsf{Spec}\, R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\mathcal{P})\}$ is the largest specialization closed subset of $\mathsf{Spec}\, R$ not containing $\mathfrak{s}(\mathcal{P})$, the second equivalence holds. The third one follows from the above discussion. As a result, $\mathsf{Supp}\, C = \mathfrak{s}(\mathfrak{s}^{-1}(\mathsf{Supp}\, C)) = \mathfrak{s}(\mathsf{BSupp}\, C)$ since $\mathfrak{s}$ is surjective.

(2) For an element $\mathfrak{p} \in \mathsf{Supp}\, C$, $\mathsf{Supp}\, C \not\subseteq \mathsf{Supp}\, \mathcal{S}(\mathfrak{p}) = \{\mathfrak{q} \in \mathsf{Spec}\, R \mid \mathfrak{q} \not\subseteq \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{p}\}$ shows $C \notin \mathcal{S}(\mathfrak{p})$. Thus, we obtain $\mathcal{S}(\mathfrak{p}) \in \mathsf{BSupp}\, C$. ■

From this lemma, the maps

$$\mathfrak{s} : \mathsf{Spc}\, \mathsf{D}^{-}(R) \rightleftarrows \mathsf{Spec}\, R : \mathcal{S}$$

restrict to maps

$$\mathfrak{s} : \mathsf{BSupp}\, C \rightleftarrows \mathsf{Supp}\, C : \mathcal{S}.$$

Lemma 3.8. *The above pair of maps induce a one-to-one correspondence*

$$\mathfrak{s} : \mathsf{Max}\, \mathsf{BSupp}\, C \rightleftarrows \mathsf{Min}\, \mathsf{Supp}\, C : \mathcal{S}.$$

Here, $\mathsf{Max}\, \mathsf{BSupp}\, C$ *(resp.* $\mathsf{Min}\, \mathsf{Supp}\, C$*) is the set of maximal (resp. minimal) elements of* $\mathsf{BSupp}\, C$ *(resp.* $\mathsf{Supp}\, C$*) with respect to the inclusion relation.*

If we take $C = R$, this lemma recovers [7, Theorem 4.12].

Proof. Because $\mathcal{S} : \mathsf{Spec}\, R \to \mathsf{Spc}\, \mathsf{D}^{-}(R)$ is injective, we have only to check that the map $\mathcal{S} : \mathsf{Min}\, \mathsf{Supp}\, C \to \mathsf{Max}\, \mathsf{BSupp}\, C$ is well-defined and surjective. Let $\mathfrak{p}$ be a minimal element of $\mathsf{Supp}\, C$. We show that $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\, C$. Take a prime thick tensor ideal $\mathcal{P}$ in $\mathsf{BSupp}\, C$ containing $\mathcal{S}(\mathfrak{p})$. Then $\mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{p}$ by Proposition 3.4. Since both $\mathfrak{p}$ and $\mathfrak{s}(\mathcal{P})$ belong to $\mathsf{Supp}\, C$ by Lemma 3.7, the minimality of $\mathfrak{p}$ shows the equality $\mathfrak{p} = \mathfrak{s}(\mathcal{P})$. Hence, we have

$$\mathsf{Supp}\, \mathcal{P} = \{\mathfrak{q} \in \mathsf{Spec}\, R \mid \mathfrak{q} \not\subseteq \mathfrak{s}(\mathcal{P}) = \mathfrak{p} = \mathfrak{s}(\mathcal{S}(\mathfrak{p}))\} = \mathsf{Supp}\, \mathcal{S}(\mathfrak{p}).$$

This shows that $\mathcal{P} \subseteq \mathcal{S}(\mathfrak{p})$ and thus $\mathcal{S}(\mathfrak{p})$ is a maximal element in $\mathsf{BSupp}\, C$. For this reason, the map $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\, C \to \mathsf{Max}\,\mathsf{BSupp}\, C$ is well-defined.

Next we check the surjectivity of the map $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\, C \to \mathsf{Max}\,\mathsf{BSupp}\, C$. Let $\mathcal{P}$ be a maximal element of $\mathsf{BSupp}\, C$. It follows from Lemma 3.7 that $\mathcal{S}\mathfrak{s}(\mathcal{P})$ is also an element in $\mathsf{BSupp}\, C$. On the other hand, $\mathcal{S}\mathfrak{s}(\mathcal{P})$ contains $\mathcal{P}$ by Proposition 3.4(3). Thus, we get $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P})$ from the maximality of $\mathcal{P}$. Let $\mathfrak{p}$ be an element of $\mathsf{Supp}\, C$ with $\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P})$. Then $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P}) \subseteq \mathcal{S}(\mathfrak{p})$. Since $\mathcal{P}$ is maximal in $\mathsf{BSupp}\, C$, one has $\mathcal{P} = \mathcal{S}(\mathfrak{p})$. Hence, $\mathfrak{p} = \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{s}(\mathcal{P})$ and this shows that $\mathfrak{s}(\mathcal{P})$ is a minimal element of $\mathsf{Supp}\, C$. As a result, one has $\mathcal{S}(\mathfrak{p}) = \mathcal{S}\mathfrak{s}(\mathcal{P}) = \mathcal{P}$ and this shows that $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\, C \to \mathsf{Max}\,\mathsf{BSupp}\, C$ is surjective. ■

The following result gives an easier way to check whether a given subspace is clopen.

Lemma 3.9. *Let X be either $\mathsf{Supp}\, C$ or $\mathsf{BSupp}\, C$ and W a subset of X. If W is both specialization closed and generalization closed, then W is clopen.*

Proof. We show this statement only for $X = \mathsf{BSupp}\, C$ because a similar argument works also for $X = \mathsf{Supp}\, C$. By symmetry, we need to check that W is closed.

Claim. $W = \bigcup_{\mathcal{P} \in \mathsf{Max}\,\mathsf{BSupp}\, C \cap W} \overline{\{\mathcal{P}\}}$.

Proof of claim. Since W is specialization closed, $W \supseteq \bigcup_{\mathcal{P} \in \mathsf{Max}\,\mathsf{BSupp}\, C \cap W} \overline{\{\mathcal{P}\}}$ holds. Let $\mathcal{P}$ be an element of W. Take a minimal element $\mathfrak{p}$ in $\mathsf{Supp}\, C$ contained in $\mathfrak{s}(\mathcal{P})$. We can take such a $\mathfrak{p}$ since $\mathsf{Supp}\, C$ is a closed subset of $\mathsf{Spec}\, R$. Then

$$\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P}) \subseteq \mathcal{S}(\mathfrak{p}).$$

By Lemma 3.8, $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\, C$. Moreover, $\mathcal{S}(\mathfrak{p})$ belongs to W since W is generalization closed and $\mathcal{P} \in W$. These show that $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\, C$. Accordingly, we obtain $\mathcal{P} \in \overline{\{\mathcal{S}(\mathfrak{p})\}}$ with $\mathcal{S}(\mathfrak{p}) \in \mathsf{Max}\,\mathsf{BSupp}\, C$ and hence the converse inclusion holds true. □

Note that $\mathsf{Supp}\, C$ is closed and thus contains only finitely many minimal elements. By using the one-to-one correspondence in Lemma 3.8, $\mathsf{Max}\,\mathsf{BSupp}\, C$ is also a finite set. Consequently, W is a finite union of closed subsets, and hence is closed. ■

Lemma 3.10. *Let U be a clopen subset of $\mathsf{BSupp}\, C$. Then*

(1) $\mathfrak{p} \in \mathfrak{s}(U)$ *if and only if* $\mathcal{S}(\mathfrak{p}) \in U$, *and*
(2) $\mathfrak{s}(U)$ *is a clopen subset in* $\mathsf{Supp}\, C$.

Proof. (1) The 'if' part is from Proposition 3.4(3). Let $\mathfrak{p}$ be an element of $\mathfrak{s}(U) \subseteq \mathfrak{s}(\mathsf{BSupp}\, C) = \mathsf{Supp}\, C$. Then there is a prime thick tensor ideal $\mathcal{P} \in U$ such that $\mathfrak{s}(\mathcal{P}) = \mathfrak{p}$. Then $\mathcal{S}(\mathfrak{p})$ belongs to U because $\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P}) = \mathcal{S}(\mathfrak{p})$ and U is generalization closed in $\mathsf{BSupp}\, C$.

(2) By Lemma 3.9, we have only to check that $\mathfrak{s}(U)$ and $\mathsf{Supp}\, C \setminus \mathfrak{s}(U)$ are specialization closed in $\mathsf{Supp}\, C$.

Take $\mathfrak{p} \in \mathfrak{s}(U)$ and $\mathfrak{q} \in \mathsf{V}(\mathfrak{p})$. Then $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})$. From (1), one has $\mathcal{S}(\mathfrak{p}) \in U$. Since U is specialization closed, we get $\mathcal{S}(\mathfrak{q}) \in U$. Thus, $\mathfrak{q} = \mathfrak{s}\mathcal{S}(\mathfrak{q})$ belongs to $\mathfrak{s}(U)$. This shows that $\mathfrak{s}(U)$ is specialization closed in $\mathsf{Supp}\, C$.

Take $\mathfrak{p} \in \mathsf{Supp}\, C \setminus \mathfrak{s}(U)$ and $\mathfrak{q} \in \mathsf{V}(\mathfrak{p})$. Then $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})$. From (1), one has $\mathcal{S}(\mathfrak{p}) \notin U$. Assume that $\mathcal{S}(\mathfrak{q})$ belongs to U. Since U is generalization closed, $\mathcal{S}(\mathfrak{p})$ belongs to U, a contradiction. Thus, $\mathcal{S}(\mathfrak{q}) \notin U$ and hence $\mathfrak{q} \notin \mathfrak{s}(U)$ by (1). This shows that $\mathsf{Supp}\, C \setminus \mathfrak{s}(U)$ is specialization closed in $\mathsf{Supp}\, C$. ■

Lemma 3.11. *Let U be a clopen subset of $\mathsf{BSupp}\, C$. Then $\mathfrak{s}^{-1}\mathfrak{s}(U) = U$.*

Proof. The inclusion $U \subseteq \mathfrak{s}^{-1}\mathfrak{s}(U)$ is trivial. For a prime thick tensor ideal $\mathcal{P} \in \mathfrak{s}^{-1}\mathfrak{s}(U)$, one has $\mathfrak{s}(\mathcal{P}) \in \mathfrak{s}(U)$. By Lemma 3.10(1), we obtain $\mathcal{S}\mathfrak{s}(\mathcal{P}) \in U$. Since U is specialization closed in $\mathsf{BSupp}\, C$ and $\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P})$, we conclude that $\mathcal{P}$ belongs U. ■

Now, we are ready to prove Theorem 3.6.

(Proof of Theorem 3.6). (1) By Lemma 3.10(2), we obtain a well-defined map

$$\{\text{clopen subsets of } \mathsf{BSupp}\, C\} \to \{\text{clopen subsets of } \mathsf{Supp}\, C\},\ U \mapsto \mathfrak{s}(U).$$

This map is injective by Lemma 3.11 and surjective since $\mathfrak{s} : \mathsf{BSupp}\, C \to \mathsf{Supp}\, C$ is continuous and surjecive. Thus, this map is an order-preserving one-to-one correspondence.

Our topological spaces $\mathsf{BSupp}\, C$ and $\mathsf{Supp}\, C$ have only finitely many connected components by Lemma 2.6, Lemma 3.8, and the proof of Lemma 3.9. Thus, connected components are nothing but minimal non-empty clopen subsets. Therefore, the statement (1) follows from the above order-isomorphism.

(2) By [1, Proposition 2.9, Proposition 2.18], every irreducible closed subset of $\mathsf{BSupp}\, C$ is of the form

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathsf{Spc}\, \mathsf{D}^{-}(R) \mid \mathcal{Q} \subseteq \mathcal{P}\}$$

for a unique prime thick tensor ideal $\mathcal{P} \in \mathsf{BSupp}\, C$. Since an irreducible component is by definition a maximal irreducible closed subset, every irreducible component of $\mathsf{BSupp}\, C$ is of the form $\overline{\{\mathcal{P}\}}$ for a unique maximal element $\mathcal{P}$ of $\mathsf{BSupp}\, C$. Thus, $\mathcal{P} = \mathcal{S}(\mathfrak{p})$ for some minimal element $\mathfrak{p}$ of $\mathsf{Supp}\, C$ by Lemma 3.8. Similarly, every irreducible component of $\mathsf{Supp}\, C$ is of the form $\overline{\{\mathfrak{p}\}}$ for a unique minimal element $\mathfrak{p}$ of $\mathsf{Supp}\, C$. Therefore, there is a maximal element $\mathcal{P}$ of $\mathsf{BSupp}\, C$ such that $\mathfrak{p} = \mathfrak{s}(\mathcal{P})$ by Lemma 3.8. Altogether, the one-to-one correspondence of Lemma 3.8 gives a one-to-one correspondence what we want. ■

The following connectedness result is a direct consequence of Theorem 3.6.

Corollary 3.12. *For a bounded complex $C \in \mathsf{D}^{\mathsf{b}}(R)$, $\mathsf{BSupp}\, C$ is connected (resp. irreducible) if and only if $\mathsf{Supp}\, C$ is connected (resp. irreducible). In particular, $\mathsf{Spc}\, \mathsf{D}^{-}(R)$ is connected (resp. irreducible) if and only if $\mathsf{Spec}\, R$ is connected (resp. irreducible).*

Remark 3.13. A part of this corollary is shown in [7, Corollary 4.13].

As an application of Theorem 3.6, we obtain the following corollary:

Corollary 3.14. *Let $C \in \mathsf{D}^{\mathsf{b}}(R)$ be a bounded complex. If C is indecomposable in $\mathsf{D}^{-}(R)$, then $\mathsf{BSupp}\, C$ is connected.*

Proof. By Theorem 3.6, it is enough to show that $\mathsf{Supp}\, C$ is connected.

Take an ideal I with $\mathsf{Supp}\, C = \mathsf{V}(I)$. It follows from [8, Lemma 2.1] that there exists a bounded complex B such that

(i) B is quasi-isomorphic to C,
(ii) $\mathsf{Supp}\, B^i \subseteq \mathsf{V}(I)$.

By (ii), we can take an integer $n > 0$ with $I^n B^i = 0$ for each i. Thus we may assume that $\mathsf{Supp}\, C = \mathsf{V}(I)$ and $IC^i = 0$ for each i.

Consider a decomposition $\mathsf{Supp}\, C = F_1 \sqcup F_2$ with F_1, F_2 closed. Then there are radical ideals I_1 and I_2 such that $F_i = \mathsf{V}(I_i)$ $(i = 1, 2)$, $I_1 + I_2 = R$, and $I_1 \cap I_2 = \sqrt{I}$. Using Chinese remainder theorem, we obtain a direct sum decomposition

$$R/\sqrt{I} \cong R/I_1 \oplus R/I_2.$$

Moreover, from the idempotent lifting theorem (see [6, Proposition 21.25]), we obtain the following decomposition

$$R/I \cong R/J_1 \oplus R/J_2.$$

Here, J_1 and J_2 are ideals with $\sqrt{J_i} = I_i$ for $i = 1, 2$. Tensoring with C, we get the following direct sum decomposition:

$$C \cong C \otimes_R R/I \cong (C \otimes_R R/J_1) \oplus (C \otimes_R R/J_2).$$

Since C is indecomposable, $C \otimes_R R/J_1 \cong C$ or $C \otimes_R R/J_2 \cong C$. If $C \otimes_R R/J_1 \cong C$, then we obtain

$$\mathsf{V}(I) = \mathsf{Supp}\, C = \mathsf{Supp}(C \otimes_R R/J_1) \subseteq \mathsf{V}(J_1)$$

and thus $\mathsf{Supp}\, C = \mathsf{V}(I) = \mathsf{V}(I_1) = F_1$. Similarly, if $C \otimes_R R/J_2 \cong C$, then one has $\mathsf{Supp}\, C = F_2$. Thus, we are done. ■

This corollary means that the Balmer support of an indecomposable R-module is connected. Such a result has been shown by Carlson [5] for the stable category of finite dimensional representations over a finite group, and more generally, by Balmer [2] for an idempotent complete rigid tensor triangulated category.

4. Realizing a clopen subset as a Balmer spectrum

In this section, we prove that every clopen subset of $\mathsf{Spc}\, \mathsf{D}^{-}(R)$ is homeomorphic to the Balmer spectrum of the Eilenberg-Moore category of some ring object. Following [3, 4], we recall the notion of a ring object and related concepts.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. We say that an object $A \in \mathcal{T}$ is a *ring object* of $\mathcal{T}$ if there is a morphisms

$$\mu : A \otimes A \to A,$$
$$\eta : \mathbf{1} \to A$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{A\otimes\mu} & A\otimes A \\ {\scriptstyle \mu\otimes A}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\otimes A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} \mathbf{1}\otimes A & \xrightarrow{\eta\otimes A} & A\otimes A & \xleftarrow{A\otimes\eta} & A\otimes \mathbf{1} \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array}$$

We say that a ring object A of $\mathcal{T}$ is *commutative* if $\mu\tau = \mu$ holds, where

$$\tau : A\otimes A \to A\otimes A$$

is the swap of factors. We say that a ring object A of $\mathcal{T}$ is *separable* if there is a morphism

$$\sigma : A \to A\otimes A$$

such that $(A\otimes\mu)(\sigma\otimes A) = \sigma\mu = (\mu\otimes A)(A\otimes\sigma)$.

We say that an object $M \in \mathcal{T}$ is a (left) A-*module* if there is a morphism

$$\lambda : A\otimes M \to M$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} A\otimes A\otimes M & \xrightarrow{A\otimes\lambda} & A\otimes M \\ {\scriptstyle \mu\otimes M}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A\otimes M & \xrightarrow[\lambda]{} & M \end{array} \qquad \begin{array}{ccc} \mathbf{1}\otimes M & \xrightarrow{\eta\otimes M} & A\otimes M \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \lambda} \\ & & M \end{array}$$

Denote by $\mathsf{Mod}\, A$ the category of A-modules.

Let me give the following easy observation.

Lemma 4.1. *If R is decomposed into $R = A\times B$ as rings, then A has a unique ring object structure by the natural multiplication $\mu : A\otimes^{\mathbf{L}}_R A \cong A\otimes_R A \cong A$ and the projection $\eta : R \to A$. Moreover, the following holds true.*

(1) *A is a commutative separable ring object in $\mathsf{D}^{-}(R)$.*
(2) *For any complex $M \in \mathsf{D}^{-}(R)$, it has an A-module structure if and only if $A\otimes^{\mathbf{L}}_R M \cong M$. This is the case, its A-module structure is uniquely determined by underlying complex.*
(3) *For A-modules M and N, $M\otimes^{\mathbf{L}}_R N$ is an A-module. Hence $\mathsf{Mod}\, A$ is a tensor triangulated category via $\otimes^{\mathbf{L}}_R$ and the forgetful functor $U_A : \mathsf{Mod}\, A \to \mathsf{D}^{-}(R)$ preserves tensor products.*

Proof. Since A is a projective R-module, the statement (1) means that A is a commutative separable R-algebra in the usual sense and this is clear. Uniqueness of this structure follows from (2).

(2) Let M be an A-module. Consider the following commutative diagram:

$$\begin{array}{ccc} R\otimes^{\mathbf{L}}_R M & \xrightarrow{\eta\otimes^{\mathbf{L}}_R M} & A\otimes^{\mathbf{L}}_R M \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \lambda} \\ & & M \end{array}$$

In particular, the composition $\mathsf{H}^i(\lambda) \circ \mathsf{H}^i(\eta \otimes^{\mathbf{L}}_R M)$ is an isomorphism for each integer i. Since $\eta \otimes^{\mathbf{L}}_R M$ is a split epimorphism, $\mathsf{H}^i(\eta \otimes^{\mathbf{L}}_R M)$ is also a split epimorphism and hence is an isomorphism for each i. This shows that $\eta \otimes^{\mathbf{L}}_R X$ is a quasi-isomorphism. From the above commutative diagram, λ is also a quasi-isomorphism.

Take an object $M \in \mathsf{D}^{-}(R)$ with $A \otimes^{\mathbf{L}}_R M \cong M$. Then the following morphism gives an A-module structure to M:

$$A \otimes^{\mathbf{L}}_R M \cong A \otimes^{\mathbf{L}}_R A \otimes^{\mathbf{L}}_R M \xrightarrow{\mu \otimes^{\mathbf{L}}_R M} A \otimes^{\mathbf{L}}_R M \cong M.$$

Moreover, the A-module structure λ is uniquely determined as

$$A \otimes^{\mathbf{L}}_R M \xrightarrow{(\eta \otimes^{\mathbf{L}}_R M)^{-1}} R \otimes^{\mathbf{L}}_R M \xrightarrow{\cong} M.$$

The last statement (3) directly follows from (2). ■

From (2) in the above lemma, we can define a unique A-module structure for a complex $M \in \mathsf{D}^{-}(R)$ with $A \otimes^{\mathbf{L}}_R M \cong M$. For simplicity, we denote this A-module by M_A. In addition, for a complex $M \in \mathsf{D}^{-}(R)$, $A \otimes^{\mathbf{L}}_R M$ has an A-module structure and hence we can define a triangulated functor

$$F_A : \mathsf{D}^{-}(R) \to \mathsf{Mod}\, A,\ M \mapsto A \otimes^{\mathbf{L}}_R M.$$

Corollary 4.2. *For any non-empty clopen subset W of $\mathsf{Spc}\,\mathsf{D}^{-}(R)$, there is a commutative separable ring object A of $\mathsf{D}^{-}(R)$ such that*

$$\varphi_A := {}^a F_A : \mathsf{Spc}(\mathsf{Mod}\, A) \to \mathsf{Spc}\,\mathsf{D}^{-}(R),\ \mathcal{P} \mapsto F_A^{-1}(\mathcal{P})$$

gives a homeomorphism onto W.

Proof. By Lemma 3.10, $\mathfrak{s}(W)$ is a clopen subset of $\mathsf{Spec}\, R$. Therefore, by Corollary 3.14, there is a direct sum decomposition $R = A \times B$ of rings with $\mathfrak{s}(W) = \mathsf{Supp}\, A$. Then Lemma 4.1 shows that A has a commutative separable ring object structure. Since U_A preserves tensor products, one can easily check that the forgetful functor $U_A : \mathsf{Mod}\, A \to \mathsf{D}^{-}(R)$ induces a continuous injective map

$$\psi_A : \mathsf{BSupp}\, A \to \mathsf{Spc}(\mathsf{Mod}\, A), \mathcal{P} \mapsto U_A^{-1}(\mathcal{P}),$$

see [1, Proposition 3.6]. Furthermore, the image of φ_A is contained in $\mathsf{BSupp}\, A$ and $\psi_A \varphi_A = 1$ because $F_A U_A \cong 1$. For this reason, we have only to check that the image of φ_A is W.

Let $\mathcal{P}$ be a prime thick tensor ideal of $\mathsf{Mod}\, A$. By definition,

$$\varphi_A(\mathcal{P}) = \{X \in \mathsf{D}^{-}(R) \mid F_A(X) = (A \otimes^{\mathbf{L}}_R X)_A \in \mathcal{P}\}$$

and it contains B because $A \otimes^{\mathbf{L}}_R B = 0$. In particular,

$$\mathsf{Supp}\, B \subseteq \mathsf{Supp}\, \varphi_A(\mathcal{P}) = \{\mathfrak{p} \in \mathsf{Spec}\, R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\varphi_A(\mathcal{P}))\}$$

and thus $\mathfrak{s}(\varphi_A(\mathcal{P})) \in \mathsf{Spec}\, R \setminus \mathsf{Supp}\, B = \mathsf{Supp}\, A$. Therefore, $\varphi_A(\mathcal{P}) \in W$ by Lemma 3.11. Conversely, take a prime thick tensor ideal $\mathcal{P}$ from W. Then $\mathfrak{s}(\mathcal{P}) \in \mathfrak{s}(W) = \mathsf{Supp}\, A$ implies that $A \notin \mathcal{P}$. Therefore,

$$\varphi_A(\psi_A(\mathcal{P})) = \{X \in \mathsf{D}^{-}(R) \mid A \otimes^{\mathbf{L}}_R X \in \mathcal{P}\} = \mathcal{P}$$

since $A \notin \mathcal{P}$. Thus, we conclude that $\varphi_A(\mathsf{Spc}(\mathsf{Mod}\, A)) = W$. ■

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