SERRE'S CONDITION FOR TENSOR PRODUCTS AND n-TOR-RIGIDITY OF MODULES

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Abstract. In this paper, we study Serre's condition (S_n) for tensor products of modules over a commutative noetherian local ring R. Especially, we consider the following question. For finitely generated R-modules M and N, either of which is $(n+1)$-Tor-rigid, if the tensor product $M \otimes_R N$ satisfies (S_{n+1}), then does $\operatorname{Tor}_i^R(M,N) = 0$ hold for all $i \geq 1$? The aim of this paper is to give an affirmative answer to this question if we assume local freeness and Serre's condition on modules. As applications, we will show that the result recovers several known results.

1. Introduction

Throughout this paper, R denotes a commutative noetherian local ring with maximal ideal $\mathfrak{m}$, and $\mathsf{X}^n(R)$ denotes the set of prime ideals $\mathfrak{p}$ of R with $\operatorname{ht} \mathfrak{p} \leq n$.

Torsion in tensor products of finitely generated R-modules has been well-studied deeply by many authors [1, 18, 19, 23]. Such a study is initiated by Auslander. The first result in this direction is the following result which is proved by Auslander for unramified regular local rings and by Lichtenbaum for general regular local rings; see [1, Lemma 3.1] and [23, Corollary 2].

Theorem 1.1 (Auslander, Lichtenbaum). *Let R be a regular local ring and let M, N be finitely generated R-modules. If $M \otimes_R N$ is torsion-free, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.*

Three decades later, Huneke and Wiegand generalized Auslander's result to hypersurface local rings, which is known as the *second rigidity theorem*; see [18, Theorem 2.7].

Theorem 1.2 (Huneke-Wiegand). *Let R be a hypersurface local ring and let M, N be finitely generated R-modules, either of which has rank. If $M \otimes_R N$ is reflexive, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.*

Recall that a finitely generated R-module M satisfies *Serre's condition* (S_n) if the inequality $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{n, \operatorname{ht} \mathfrak{p}\}$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$. We note that a non-zero finitely generated R-module M is torsion-free if and only if it satisfies (S_1) over a regular local ring and that M is reflexive if and only if it satisfies (S_2) over a hypersurface local ring. From this observation, Huneke, Jorgensen, and Wiegand [17] asked the following question.

Question 1.3 (Huneke-Wiegand-Jorgensen). Let R be a complete intersection local ring of codimension c and M, N finitely generated R-modules. If $M \otimes_R N$ satisfies (S_{c+1}), then does $\operatorname{Tor}_i^R(M,N) = 0$ hold for all $i \geq 1$?

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An answer to this question is given by Dao [13, Theorem 7.6] under some extra assumptions:

Theorem 1.4 (Dao)**.** *Let R be a complete intersection local ring of codimension c whose completion is a quotient of an unramified regular local ring. Let M and N be finitely generated R-modules. Assume that the following conditions hold;*

(1) *$N_{\mathfrak{p}}$ is free for $\mathfrak{p} \in \mathsf{X}^c(R)$.*
(2) *M and N satisfy (S_c).*

If $M \otimes_R N$ satisfies (S_{c+1}), then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The key role of the proof of Theorem 1.4 is played by *n-Tor-rigidity* of modules; see Definition 2.4. Indeed, the proof relies upon the fact that the vanishing of the *eta pairing*, which is a generalization of *Hochster's theta pairing*, implies that c-Tor-rigidity of a pair of finitely generated R-modules. The unramified assumption on an embedded regular local ring is used here.

On the other hand, it is shown by Murthy [22] that over a complete intersection local ring R of codimension c, every finitely generated R-module is $(c+1)$-Tor-rigid. In this paper, we consider another direction of a variant of Theorems 1.1 and 1.2. Namely, we consider the following question.

Question 1.5. Let R be a noetherian local ring and M, N finitely generated R-modules and n a non-negative integer. If $M \otimes_R N$ satisfies (S_{n+1}) and either M or N is $(n+1)$-Tor-rigid, then does $\operatorname{Tor}_i^R(M, N) = 0$ hold for all $i \geq 1$?

This paper aims to answer this question. Namely, we prove the following result.

Theorem 1.6 (see Theorem 3.4)**.** *Let n be a non-negative integer and let R be a noetherian local ring satisfying (S_n) when $n \geq 1$ and (S_1) when $n = 0$. Let M and N be finitely generated R-modules. Assume that the following conditions hold;*

(1) *N satisfies (S_n) and $N_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \mathsf{X}^n(R)$.*
(2) *M is stably isomorphic to the nth syzygy of an $(n+1)$-Tor-rigid module.*

If $M \otimes_R N$ satisfies (S_{n+1}), then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

Here, we say that two finitely generated R-modules M and N are *stably isomorphic* if there are finitely generated free R-modules F and G such that $M \oplus F \cong N \oplus G$.

If we assume R is a complete intersection local ring of codimension c, then every finitely generated R-module is $(c+1)$-Tor-rigid and that a finitely generated R-module is the cth syzygy of some finitely generated R-module if and only if it satisfies (S_c). Thus, this result recovers Theorem 1.4. Moreover, we want to emphasize that the unramified assumption on an embedded regular local ring is removed in our theorem.

The organization of this paper is as follows. Section 2 is devoted to the preparation of the proof. In Section 3, we will prove Theorem 3.4 and give several applications of the main theorem, which recovers some known results such as Theorem 1.4.

2. Preliminaries.

In this section, we recall several basic definitions, including that of an n-Tor-rigid module, for later use.

Definition 2.1. ([4]) Let M be a finitely generated R-module. A *(codimension c) quasi-deformation* of R is a diagram $R \to R' \twoheadleftarrow Q$ of local rings such that $R \to R'$ is flat and $R' \twoheadleftarrow Q$ is surjective with kernel generated by a Q-regular sequence (of length c). The *complete intersection dimension* of M is defined to be

$$\text{CI-dim}_R(M) := \inf\{\text{pd}_Q(M \otimes_R R') - \text{pd}_Q R' \mid R \to R' \twoheadleftarrow Q \text{ is a quasi-deformation of } M\}.$$

Definition 2.2. Let M be a finitely generated R-module. The *complexity* of M is

$$\text{cx}_R(M) := \inf\{c \geq 0 \mid \text{there exists } r > 0 \text{ such that } \beta_i(M) \leq ri^{c-1} \text{ for all } i \gg 0\}.$$

Here, $\beta_i(M) := \dim_k \operatorname{Ext}^i_R(M, k)$ denotes the ith *Betti number* of M.

For the basic properties of complete intersection dimension and complexity, we refer the reader to [2, 3, 4]. We only record the following important fact on complete intersection dimension and complexity.

Proposition 2.3 ([4, Theorem 1.3] and [16, Theorem 2.3])**.** *The following are equivalent for a noetherian local ring R;*

(1) *R is complete intersection.*
(2) $\text{CI-dim}_R(M) < \infty$ *for every finitely generated R-module M.*
(3) $\text{cx}_R(M) < \infty$ *for every finitely generated R-module M.*

If this is the case, $\text{cx}_R(M) \leq \operatorname{codim} R$ *for every finitely generated R-module M.*

Here, we give the definition of an n-Tor-rigid module.

Definition 2.4. Let n be a positive integer. We say that a finitely generated R-module M is *n-Tor-rigid* if for all finitely generated R-modules N and positive integers t, the n-consecutive vanishing $\operatorname{Tor}_i^R(M, N) = 0$ for $i = t, t+1, \dots, t+n-1$ implies that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq t$.

There are various examples of n-Tor-rigid modules in the literature.

Example 2.5. (a) ([22, Corollary 1.9]) Assume that R is a complete intersection local ring of codimension c. Then every finitely generated R-module is $(c+1)$-Tor-rigid.
(b) ([13, Corollary 6.8]) Let R be a local complete intersection ring of codimension $c > 0$ whose completion is a quotient of an unramified regular local ring. Then a finitely generated R-module M is c-Tor-rigid if $\text{cx}_R(M) < c$.
(c) (cf. [8, Corollary 4.3] and [20, Proposition 2.3]) Let M be a finitely generated R-module. If $\text{CI-dim}_R(M) = 0$ and $\text{cx}_R(M) = c$, then M is $(c+1)$-Tor-rigid.
(d) ([6, Theorem 5(ii)]) Let I be an ideal satisfying $\mathfrak{m}I \neq \mathfrak{m}(I : \mathfrak{m})$. Then R/I and hence I is 2-Tor-rigid.
(e) ([21, Lemma, page 316]) Let M be a finitely generated R-module with $\mathfrak{m}M \neq 0$. Then $\mathfrak{m}M$ is 2-Tor-rigid.

We introduce the following fact, which is essentially shown by Auslander [1, Lemma 3.1]. It is a prototype for our main result.

Proposition 2.6. *Let R be a noetherian local ring and M, N finitely generated R-modules. Assume that the following conditions hold;*

(1) *N has rank, i.e., there is a nonnegative integer r such that $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus r}$ for all associated prime ideals $\mathfrak{p}$ of R.*
(2) *M is 1-Tor-rigid.*

If $M \otimes_R N$ satisfies (S_1), then $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.

Proof. Consider the short exact sequence $0 \to \top N \to N \to \bot N \to 0$ where $\top N$ and $\bot N$ are the torsion part and the torsion-free part of N, respectively. Applying $M \otimes_R -$ to this sequence, we obtain an exact sequence

$$\mathrm{Tor}_1^R(M, \bot N) \to M \otimes_R \top N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{\beta} M \otimes_R \bot N \to 0.$$

As $M \otimes_R \top N$ is torsion and $M \otimes_R N$ is torsion-free, the map α must be zero and thus β is an isomorphism. If it is shown that $\mathrm{Tor}_1^R(M, \bot N) = 0$, then $M \otimes_R \top N = 0$ and hence $M = 0$ or $\top N = 0$. Therefore we may assume that N is torsion-free.

Since N is a torsion-free R-module having a rank, we can take a short exact sequence

$$0 \to N \to F \to C \to 0$$

with F free and C torsion. Tensoring with M, we obtain an exact sequence

$$0 \to \mathrm{Tor}_1^R(M, C) \to M \otimes_R N \to M \otimes_R F \to M \otimes_R C \to 0.$$

Since $\mathrm{Tor}_1^R(M, C)$ is torsion and $M \otimes_R N$ is torsion-free, we see that $\mathrm{Tor}_1^R(M, C) = 0$. Then the 1-Tor-rigidity of N shows $\mathrm{Tor}_i^R(M, C) = 0$ for all $i \geq 1$ and in particular $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. ■

3. Main theorem and its applications

The purpose of this section is to prove our main theorem. Before beginning the proof, we establish several lemmas.

Lemma 3.1. *Let M and N be finitely generated R-modules, and $\underline{x} := x_1, \dots, x_n$ be an N-regular sequence. If $\mathrm{Tor}_1^R(M, N/\underline{x}N) = 0$, then $\mathrm{Tor}_1^R(M, N) = 0$. The converse holds if moreover, $\underline{x}$ is regular on $M \otimes_R N$.*

Proof. The first statement follows easily from Nakayama's lemma.

Assume that $\underline{x}$ is regular on both $M \otimes_R N$ and N and that $\mathrm{Tor}_1^R(M, N) = 0$. From the short exact sequence $0 \to N \xrightarrow{x_1} N \to N/x_1N \to 0$, we obtain an exact sequence

$$\mathrm{Tor}_1^R(M, N) = 0 \to \mathrm{Tor}_1^R(M, N/x_1N) \to M \otimes_R N \xrightarrow{x_1} M \otimes_R N \to M \otimes_R (N/x_1N) \to 0.$$

Then we have $\mathrm{Tor}_1^R(M, N/x_1N) = 0$ since x_1 is regular on $M \otimes_R N$. On the other hand, the isomorphism $M \otimes_R (N/x_1N) \cong (M \otimes_R N)/x_1(M \otimes_R N)$ shows that the sequence $x_2, \dots, x_n$ is regular on both $M \otimes_R (N/x_1N)$ and N/x_1N. Therefore the induction argument on n shows that $\mathrm{Tor}_1^R(M, N/\underline{x}N) = 0$. ■

Lemma 3.2. *Let M and N be finitely generated R-modules, and $\underline{x} := x_1, \dots, x_n$ an N-regular sequence such that $\underline{x}\mathrm{Tor}_i^R(M, N) = 0$ for $i = 1, \dots, n$. If $\mathrm{Tor}_{n+1}^R(M, N/\underline{x}^{2^{n-1}}N) = 0$, then $\mathrm{Tor}_i^R(M, N) = 0$ for $i = 1, 2, \dots, n+1$. Here, $\underline{x}^{2^{n-1}}$ denotes the sequence $x_1^{2^{n-1}}, \dots, x_n^{2^{n-1}}$.*

Proof. Notice that $\operatorname{Tor}_{n+1}^R(M,N)=0$ holds by Lemma 3.1. We prove $\operatorname{Tor}_i^R(M,N)=0$ for $i=1,\dots,n+1$ by induction on n.

Suppose that $n=1$. Consider the following pullback diagram of $N \xrightarrow{x_1} N$ and a free cover $F \to N$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & N/x_1N & = & N/x_1N & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega N & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0\\
 & & \| & & \uparrow & \text{PB} & \uparrow{\scriptstyle x_1} & & \\
0 & \longrightarrow & \Omega N & \longrightarrow & W & \longrightarrow & N & \longrightarrow & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then the middle column means that W is the first syzygy of N/x_1N i.e., $W=\Omega(N/x_1N)$. Applying $M\otimes_R -$ to the middle and the bottom rows, we get a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Tor}_1^R(M,F)=0 & \longrightarrow & \operatorname{Tor}_1^R(M,N) & \longrightarrow & M\otimes_R \Omega N\\
\uparrow & & {\scriptstyle x_1}\uparrow & & \|\\
\operatorname{Tor}_1^R(M,\Omega(N/x_1N))=0 & \longrightarrow & \operatorname{Tor}_1^R(M,N) & \longrightarrow & M\otimes_R \Omega N.
\end{array}$$

Thus the assumption $x_1\operatorname{Tor}_1^R(M,N)=0$ implies that $\operatorname{Tor}_1^R(M,N)=0$.

Now let $n\geq 1$ and set $N_k:=N/(x_1^{2^{n-1}},\dots,x_k^{2^{n-1}})N$ for $k=0,\dots,n$. First we show:

Claim. $\underline{x}^{2^k}\operatorname{Tor}_i^R(M,N_k)=0$ for $k+1\leq i\leq n$.

Proof of Claim. Let us proceed by induction on k. If $k=0$, the claim follows by the assumption. Consider the case of $k\geq 1$ and assume $\underline{x}^{2^{k-1}}\operatorname{Tor}_i^R(M,N_{k-1})=0$ for $k\leq i\leq n$. The short exact sequence $0\to N_{k-1}\xrightarrow{x_k^{2^{n-1}}} N_{k-1}\to N_k\to 0$ induces the exact sequence

$$\operatorname{Tor}_i^R(M,N_{k-1})\xrightarrow{\varphi}\operatorname{Tor}_i^R(M,N_k)\xrightarrow{\psi}\operatorname{Tor}_{i-1}^R(M,N_{k-1})$$

of R-modules. Let $k+1\leq i\leq n$ be an integer and let $j=1,2,\dots,n$. For any element $\xi\in\operatorname{Tor}_i^R(M,N_k)$, one has $\psi(x_j^{2^{k-1}}\xi)=x_j^{2^{k-1}}\psi(\xi)=0$ as $x_j^{2^{k-1}}$ kills $\operatorname{Tor}_{i-1}^R(M,N_{k-1})$. The exactness of the above sequence shows that there is an element $\zeta\in\operatorname{Tor}_i^R(M,N_{k-1})$ with $\varphi(\zeta)=x_j^{2^{k-1}}\xi$. Again using the induction hypothesis $x_j^{2^{k-1}}\operatorname{Tor}_i^R(M,N_{k-1})=0$, we conclude that $x_j^{2^k}\xi=x_j^{2^{k-1}}(x_j^{2^{k-1}}\xi)=x_j^{2^{k-1}}\varphi(\zeta)=\varphi(x_j^{2^{k-1}}\zeta)=0$. Hence we conclude that $\underline{x}^{2^k}\operatorname{Tor}_i(M,N_k)=0$ for $k+1\leq i\leq n$. □

It follows by the claim, we have $\underline{x}^2\operatorname{Tor}_i^R(\Omega M,N_1)=0$ for $1\leq i\leq n-1$. On the other hand, there are isomorphisms

$$\operatorname{Tor}_n^R(\Omega M,N_1/(x_2^{2^{n-1}},\dots,x_n^{2^{n-1}})N_1)\cong\operatorname{Tor}_{n+1}^R(M,N/\underline{x}^{2^{n-1}}N)\cong 0$$

from the assumption. Therefore, the induction hypothesis shows $\operatorname{Tor}_i^R(\Omega M, N_1) = 0$ for $1 \leq i \leq n$. By Lemma 3.1, we get $\operatorname{Tor}_i^R(M, N) = 0$ for $2 \leq i \leq n+1$. Finally, from $\operatorname{Tor}_2^R(M, N_1) = 0$ and $x_1^2 \operatorname{Tor}_i^R(M, N) = 0$ for $i = 1, 2$, we also obtain $\operatorname{Tor}_1^R(M, N) = 0$ as the base step of the induction. ■

Recall that the *codimension* of a finitely generated R-module M is defined to be

$$\operatorname{codim} M := \inf\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}.$$

From the definition, $\operatorname{codim} M \geq n$ if and only if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathsf{X}^n(R)$.

Lemma 3.3. *Let M and N be finitely generated R-modules such that $\operatorname{CI-dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathsf{X}^n(R)$. If $\operatorname{codim} \operatorname{Tor}_i^R(M, N) \geq n+1$ for all $i \gg 1$, then $\operatorname{codim} \operatorname{Tor}_i^R(M, N) \geq n+1$ for all $i \geq 1$.*

Proof. Fix $\mathfrak{p} \in \mathsf{X}^n(R)$ and it suffices to show that $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for all $i \geq 1$. From the assumption we get $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for all $i \gg 1$ and this implies $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for all $i \geq 1$ as $\operatorname{CI-dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = 0$; see [3, Theorem 4.9]. ■

Now we are ready to prove our main theorem. It is a quite general extension of Proposition 2.6.

Theorem 3.4. *Let n be a non-negative integer, and let R be a noetherian local ring satisfying (S_n) when $n \geq 1$ and (S_1) when $n = 0$. Let M and N be finitely generated R-modules. Assume that the following conditions hold;*

(1) *N satisfies (S_n) and $\operatorname{CI-dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \infty$ for $\mathfrak{p} \in \mathsf{X}^n(R)$.*
(2) *M is stably isomorphic to the nth syzygy of an $(n+1)$-Tor-rigid module M',*
(3) *$\operatorname{codim} \operatorname{Tor}_i^R(M, N) \geq n+1$ for all $i \gg 1$,*

If $M \otimes_R N$ satisfies (S_{n+1}), then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

Proof. If $n \geq \dim R$, then there is nothing to prove by the assumption (3). We may assume that $n < \dim R$.

First, consider the case of $n = 0$. Using the same argument as in the proof of Proposition 2.6, we may assume N is torsion-free and hence it satisfies (S_1) because so does R. It follows from [15, Proposition 2.4] and the assumption (1) that there is a short exact sequence $0 \to N \to F \to C \to 0$ with F free. Then the same proof as in Proposition 2.6 works and we complete the proof in this case.

Next, let us consider the case of $n \geq 1$. Again using [15, Proposition 2.4], there is a short exact sequence

$$(*) \qquad 0 \to N \to F \to C \to 0$$

with F free, C satisfies (S_{n-1}), and $\operatorname{Ext}_R^1(C, R) = 0$. Since N satisfies (S_n), one has $\operatorname{CI-dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathsf{X}^n(R)$. Thus, $\operatorname{Ext}_{R_{\mathfrak{p}}}^1(C_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ shows that $\operatorname{CI-dim}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathsf{X}^n(R)$ by [12, Lemma 1.1.10] and [4, Theorem 1.4]. It follows by Lemma 3.3, we have $\operatorname{codim} \operatorname{Tor}_i(M', C) \geq n+1$ for all $i \geq 1$. In particular, $\operatorname{Tor}_i(M', C)$ are torsion for all $i \geq 1$. Tensoring M with the above sequence $(*)$, we get an exact sequence

$$0 \to \operatorname{Tor}_1^R(M, C) \to M \otimes_R N \to M \otimes_R F \to M \otimes_R C \to 0.$$

Since $M\otimes_R N$ satisfies (S_{n+1}) and $\operatorname{Tor}_1^R(M,C)\cong\operatorname{Tor}_{n+1}^R(M',C)$ is torsion, we get $\operatorname{Tor}_1^R(M,C)=0$.

Next, we prove that $\operatorname{Tor}_i^R(M',C)=0$ for $i=2,\dots,n+1$. To this end, we may assume $n\geq 2$ since the case of $n=1$ follows by $\operatorname{Tor}_2^R(M',C)\cong\operatorname{Tor}_1^R(M,C)=0$. Set $\mathfrak{a}:=\bigcap_{1\leq i\leq n}\operatorname{ann}\operatorname{Tor}_i^R(M',C)$. It follows from $\operatorname{codim}\operatorname{Tor}_i^R(M',C)\geq n+1$ for all $i\geq 1$ that $\mathrm{V}(\mathfrak{a})$ consists only prime ideals of height at least $n+1$. Using (S_{n-1}) condition on C, one has the following (in)equalities:

$$\operatorname{grade}(\mathfrak{a},C)=\inf\{\operatorname{depth}_{R_\mathfrak{p}}C_\mathfrak{p}\mid\mathfrak{p}\in V(\mathfrak{a})\}\geq\inf\{\operatorname{depth}_{R_\mathfrak{p}}C_\mathfrak{p}\mid\operatorname{ht}\mathfrak{p}\geq n+1\}\geq n-1.$$

Here, the first equality uses [5, Proposition 1.2.10(a)]. On the other hand, for a prime ideal $\mathfrak{p}$ with height at least $n+1$, the depth lemma [5, Proposition 1.2.9] applying to the short exact sequence

$$0\to(M\otimes_R N)_\mathfrak{p}\to(M\otimes_R F)_\mathfrak{p}\to(M\otimes_R C)_\mathfrak{p}\to 0$$

implies that $\operatorname{depth}_{R_\mathfrak{p}}(M\otimes_R C)_\mathfrak{p}\geq n$. Thus the similar argument as above shows that $\operatorname{grade}(\mathfrak{a},M\otimes_R C)\geq n$. Therefore we can take a sequence $\underline{x}:=x_1,\dots,x_{n-1}$ of elements from $\mathfrak{a}$ which is regular on both C and $M\otimes_R C$. Then we get $\operatorname{Tor}_n^R(\Omega M',C/\underline{x}^{2^{n-2}}C)\cong\operatorname{Tor}_1^R(M,C/\underline{x}^{2^{n-2}}C)=0$ by Lemma 3.1. Because $\underline{x}$ is taken from the annihilators of $\operatorname{Tor}_i^R(\Omega M',C)\cong\operatorname{Tor}_{i+1}^R(M',C)$ for $1\leq i\leq n-1$, Lemma 3.2 gives us that $\operatorname{Tor}_i^R(M',C)\cong\operatorname{Tor}_{i-1}^R(\Omega M',C)=0$ for $2\leq i\leq n+1$.

To use the $(n+1)$-Tor-rigidity of M', it remains to show $\operatorname{Tor}_1^R(M',C)=0$. If this is true, the vanishing $\operatorname{Tor}_i^R(M',C)=0$ for $1\leq i\leq n+1$ implies that $\operatorname{Tor}_i^R(M',C)=0$ for all $i\geq 1$ and hence we conclude that $\operatorname{Tor}_i^R(M,N)=0$ for all $i\geq 1$. Assume the contrary that $\operatorname{Tor}_1^R(M',C)\neq 0$ and take $\mathfrak{p}\in\operatorname{Ass}\operatorname{Tor}_1^R(M',C)$. Notice that $\operatorname{ht}\mathfrak{p}\geq n+1$ as $\operatorname{codim}\operatorname{Tor}_1^R(M',C)\geq n+1$. Let $F.$ be a free resolution of M' and decompose it into short exact sequences $0\to\Omega^{k+1}M'\to F_k\to\Omega^k M'\to 0$ for $k\geq 0$. Tensoring these sequences with C, we obtain the following exact sequences

$$\begin{aligned}0\to\operatorname{Tor}_1^R(M',C)\to\Omega M'\otimes_R C\to &F_0\otimes_R C\to M'\otimes_R C\to 0\\ 0\to\Omega^2M'\otimes_R C\to &F_1\otimes_R C\to\Omega M'\otimes_R C\to 0\\ &\cdots\\ 0\to M\otimes_R C\to &F_{n-1}\otimes_R C\to\Omega^{n-1}M'\otimes_R C\to 0.\end{aligned}$$

Here we use $\operatorname{Tor}_1^R(\Omega^iM',C)\cong\operatorname{Tor}_{i+1}^R(M',C)=0$ for $1\leq i\leq n$. It has been already explained above that $\operatorname{depth}_{R_\mathfrak{p}}C_\mathfrak{p}\geq n$ and $\operatorname{depth}_{R_\mathfrak{p}}(M\otimes_R C)_\mathfrak{p}\geq n$ because $\operatorname{ht}\mathfrak{p}\geq n+1$. Applying the depth lemma [5, Proposition 1.2.9] to the above sequences, we obtain $\operatorname{depth}_{R_\mathfrak{p}}(\Omega M'\otimes_R C)_\mathfrak{p}\geq 1$ This contradicts to $\mathfrak{p}\in\operatorname{Ass}\operatorname{Tor}_1^R(M',C)$. Thus we conclude that $\operatorname{Tor}_1^R(M',C)=0$ and this finishes the proof. ■

Example 3.5. Let k be a field, $R:=k[[x,y]]/(xy)$ and $M:=R/(x)$. Note that M is maximal Cohen-Macaulay having finite complete intersection dimension. Moreover M is the first syzygy of an 2-Tor-rigid module $R/(y)$. Then $M\otimes_R M\cong M$ satisfies (S_2) but $\operatorname{Tor}_1^R(M,M)\cong k\neq 0$. This explains that the condition (3) in Theorem 3.4 is necessary. Indeed, $\operatorname{codim}\operatorname{Tor}_i^R(M,M)=1$ for all odd i.

For the rest of this paper, we will give several applications of the main theorem using n-Tor-rigid modules that appeared in Example 2.5.

First, consider the case of $n = c$ for complete intersection local rings of codimension c. As we have explained in the introduction that this case of Theorem 3.4 recovers Dao's result.

Corollary 3.6 (cf. [13, Theorem 7.6])**.** *Let R be a complete intersection local ring of codimension c and let M, N be finitely generated R-modules. Assume that the following conditions hold;*
(a) *M and N satisfy (S_c).*
(b) $\operatorname{codim} \operatorname{Tor}_i^R(M, N) \geq c + 1$ *for all $i \gg 0$.*
If $M \otimes_R N$ satisfies (S_{c+1}), then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

Proof. The condition (1) in Theorem 3.4 follows from [4, Proposition 1.6]. Since R is a complete intersection, the assumption (a) shows that M is (stably) isomorphic to the $(c + 1)$st syzygy of some finitely generated R-module M'. It follows from [22, Corollary 1.9] that M' is $(c + 1)$-Tor-rigid module. Thus, the condition (2) in Theorem 3.4 is satisfied. ■

For the case of $n = c - 1$ over a complete intersection local ring, the main theorem also recovers the result by Celikbas.

Corollary 3.7 (cf. [7, Theorem 3.4])**.** *Let R be a complete intersection local ring of codimension $c \geq 1$ whose completion is a quotient of an unramified regular local ring. Let M and N be finitely generated R-modules. Assume that the following conditions hold;*
(a) *M and N satisfy (S_{c-1}).*
(b) $\operatorname{codim} \operatorname{Tor}_i^R(M, N) \geq c$ *for all $i \gg 0$.*
If $M \otimes_R N$ satisfies (S_c), then $\operatorname{cx}_R(M) = \operatorname{cx}_R(N) = c$ or $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

Proof. If $\operatorname{cx}_R(M) < c$ or $\operatorname{cx}_R(N) < c$, then M or N is c-Tor-rigid by [13, Corollary 6.8]. Thus, using the similar argument as in the proof of Corollary 3.6, we can apply Theorem 3.4 for $n = c - 1$. ■

Recall that an ideal I of R is called *Burch* if it satisfies $\mathfrak{m}I \neq \mathfrak{m}(I : \mathfrak{m})$. Examples of Burch ideals are non-zero ideals of the form $\mathfrak{m}J$, or integrally closed ideals, or $\mathfrak{m}$-primary and weakly $\mathfrak{m}$-full ideals; see [14]. It has been shown in [6, Theorem 5(ii)] that R/I is a 2-Tor-rigid module for a Burch ideal I. Thus, we can apply our main theorem to the module I. The next result shows that Burch ideals test finite projective dimension of modules, which should be compared with [10, Corollary 2.7] and [11, page 842].

Corollary 3.8. *Let R be a noetherian local ring satisfying (S_1), I a Burch ideal, and N a finitely generated R-module. Assume that the following conditions hold;*
(a) *N satisfies (S_1) and $\text{CI-dim}\, N_\mathfrak{p} < \infty$ for $\mathfrak{p} \in \mathsf{X}^1(R)$.*
(b) $\operatorname{ht} I \geq 2$.
If $I \otimes_R N$ satisfies (S_2), then $\operatorname{pd}_R N \leq 2$.

Proof. We apply Theorem 3.4 to R-modules N and $M := I$. Since $\operatorname{ht} I \geq 2$, I is locally free on $\mathsf{X}^1(R)$ and hence the condition (3) in Theorem 3.4 is satisfied. The condition (2) follows from [6, Theorem 5(ii)]. Thus, Theorem 3.4 shows that $\operatorname{Tor}_{i+1}^R(R/I, N) \cong \operatorname{Tor}_i^R(I, N) = 0$ for $i \geq 1$. Again using [6, Theorem 5(ii)], we conclude that $\operatorname{pd}_R N \leq 2$. ■

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