ON AN EXAMPLE CONCERNING THE SECOND RIGIDITY THEOREM

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ABSTRACT. In this paper we revisit an example of Celikbas and Takahashi concerning the reflexivity of tensor products of modules. We study Tor-rigidity and the Hochster–Huneke graph with vertices consisting of minimal prime ideals, and determine a condition with which the aforementioned example cannot occur. Our result, in particular, corroborates the Second Rigidity Theorem of Huneke and Wiegand.

1. INTRODUCTION

Throughout R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$, and all R-modules are assumed to be finitely generated. For unexplained notations and terminology, such as the definitions of homological dimensions, we refer the reader to [4, 6, 20].

In this paper we are concerned with the following result of Huneke and Wiegand, which is known as the *Second Rigidity Theorem*; see [16, 2.1].

Theorem 1.1. (*Huneke and Wiegand* [16]) *Let R be a hypersurface ring, and let M and N be R-modules such that M has rank, i.e., there is a nonnegative integer r such that $M_{\mathfrak{p}}$ is free of rank r for each associated prime ideal $\mathfrak{p}$ of R (e.g., $\mathsf{pd}_R(M) < \infty$). If $M \otimes_R N$ is reflexive, or in this context equivalently, is a second syzygy module, then N is reflexive.* □

Another conclusion of Theorem 1.1, which is worth noting, is the vanishing of $\mathsf{Tor}_i^R(M,N)$ for each $i \geq 1$. For quite some time it has been an open problem whether the module M in Theorem 1.1 must also be reflexive; see [18]. Recently Celikbas and Takahashi [10] has given an example disproving this query: there is a reduced hypersurface ring R, and modules M and N over R such that both $M \otimes_R N$ and N are reflexive, $\mathsf{pd}_R(M) < \infty$, but M is not reflexive. Moreover, it can be easily checked that there exists a prime ideal $\mathfrak{q}$ of R of height one such that the module N in the example satisfies $\mathsf{pd}_{R_{\mathfrak{q}}}(N_{\mathfrak{q}}) = \infty$; see Example 4.5 for details. The main aim of this paper is to show that such an example cannot occur in case $\mathsf{pd}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \infty$ for each prime ideal $\mathfrak{p}$ of R of height at most one. More precisely, we prove:

Theorem 1.2. *Assume R is a hypersurface ring (quotient of an unramified regular local ring), and M and N are nonzero R-modules. Assume further:*

(i) $\mathsf{pd}(M) < \infty$.
(ii) $\mathsf{pd}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \infty$ *for each prime ideal $\mathfrak{p}$ of R of height at most one.*

If $M \otimes_R N$ is reflexive, then both M and N are reflexive.

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We give a proof of Theorem 1.2 in section 4, but in fact our main argument is more general: we consider tensor products $M \otimes_R N$ which are n-th syzygy modules for $n \geq 2$ and modules N of finite complete intersection dimension over rings that are not necessarily hypersurfaces; see Theorem 3.1. A key ingredient of our proof is the fact that, when R satisfies Serre's condition (S_2), the *Hochster-Huneke graph* [15] is connected; see Theorem 4.3.

2. Preliminaries

2.1. An R-module M is said to be *Tor-rigid* provided that the following condition holds: if N is an R-module with $\mathsf{Tor}_1^R(M,N)=0$, then $\mathsf{Tor}_2^R(M,N)=0$. Examples of Tor-rigid modules are abundant in the literature. For example, each syzgy of the R-module M is Tor-rigid if:

(i) R is a hypersurface that is quotient of an unramified regular local ring, and M has either finite length or finite projective dimension; see [16, 2.4] and [19, Theorem 3].
(ii) R has positive depth and $M = \mathfrak{m}^r$ for some integer $r \geq 1$; see [11, 2.5]. □

2.2. Let M be an R-module with a projective presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$. The *transpose* $\mathsf{Tr}M$ of M is the cokernel of $f^* = \mathsf{Hom}_R(f,R)$, and hence is given by the exact sequence: $0 \to M^* \to P_0^* \to P_1^* \to \mathsf{Tr}M \to 0$. Note $\mathsf{Tr}M$ is well-defined up to projective summands.

Given an integer $n \geq 0$, it follows from [2, 2.8] that there is an exact sequence of functors:

$$0 \to \mathsf{Ext}_R^1(\mathsf{Tr}\Omega^n M,-) \to \mathsf{Tor}_n^R(M,-) \to \mathsf{Hom}_R(\mathsf{Ext}_R^n(M,R),-) \to \mathsf{Ext}_R^2(\mathsf{Tr}\Omega^n M,-). \qquad \square$$

Recall that an R-module N is said to be *torsionless* if the natural map $N \to N^{**}$ is injective, i.e., $\mathsf{Ext}_R^1(\mathsf{Tr}N,R)=0$; see 2.2.

2.3. Let N be a torsionless R-module and let $\{f_1,f_2,\dots,f_s\}$ be a minimal generating set of the module $N^* = \mathsf{Hom}(N,R)$. Let $\delta : R^{\oplus s} \twoheadrightarrow N^*$ be defined by $\delta(e_i) = f_i$ for $i = 1,2,\dots,s$, where $\{e_1,e_2,\dots,e_s\}$ is the standard basis for $R^{\oplus s}$. Then, composing the natural injective map $N \hookrightarrow N^{**}$ with δ^*, we obtain the short exact sequence:

$$0 \to N \xrightarrow{u} R^{\oplus s} \to N_1 \to 0,$$

where $u(x) = (f_1(x), f_2(x), \dots, f_s(x))$ for all $x \in N$; see 2.2. Any module N_1 obtained in this way is called a *pushforward* (or *left projective approximation*) of M; see [3, 13]. Note that such a construction is unique, up to a non-canonical isomorphism; see, for example, [13, page 62]. Also it follows $\mathsf{Ext}_R^1(N_1,R)=0$ so that $\Omega\mathsf{Tr}N \cong \mathsf{Tr}N_1$ (up to free summands); see [2, 3.9]. □

2.4. Let M be an R-module and let $n \geq 0$ be an integer. Then M is said to satisfy $(\widetilde{S_n})$ provided $\mathsf{depth}_{R_\mathfrak{q}}(M_\mathfrak{q}) \geq \min\{n, \mathsf{depth}(R_\mathfrak{q})\}$ for each $\mathfrak{q} \in \operatorname{Supp}(M)$ (note $\mathsf{depth}(0)=\infty$) If R is Cohen-Macaulay, then M satisfies $(\widetilde{S_n})$ if and only if M satisfies *Serre's condition* (S_n); see [13]. □

2.5. Given an integer $s \geq 0$, we set $\mathsf{Y}^s(R) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathsf{depth}(R_\mathfrak{p}) \leq s\}$. In particular, $\mathsf{Y}^0(R)$ denotes the set of all *associated* prime ideals of R.

2.6. ([12, 2.4] and [13, 3.8]) Let M be an R-module and let $n \geq 1$ be an integer. Assume that $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for each $\mathfrak{p} \in \mathsf{Y}^{n-1}(R)$. Then the following conditions are equivalent:

(i) M satisfies $(\widetilde{S_n})$.
(ii) M is *n-torsion-free*, i.e., $\mathsf{Ext}_R^i(\mathsf{Tr}M,R)=0$ for each $i = 1,\dots,n$.
(iii) M is an *n-th syzygy* module, i.e., $M \cong \Omega^n(N)$ for some R-module N. □

2.7. Let M and N be R-modules with $\mathsf{CI\text{-}dim}(M) < \infty$ or $\mathsf{CI\text{-}dim}(N) < \infty$. If $\mathsf{Tor}_i^R(M,N) = 0$ for each $i \geq 1$, then $\mathsf{depth}(M) + \mathsf{depth}(N) = \mathsf{depth}(R) + \mathsf{depth}(M \otimes_R N)$, i.e., the *depth formula* holds; see [1, 2.5]. □

2.8. Let M and N be R-modules such that $\mathsf{CI\text{-}dim}(N) = 0$. Then $\mathsf{Ext}_R^i(\mathsf{Tr}N, M) = 0$ for all $i \geq 1$ if and only if $\mathsf{Tor}_i^R(M,N) = 0$ for all $i \geq 1$; see [9, 3.2]. □

3. MAIN THEOREM

In this section we will prove the following theorem which is our main result:

Theorem 3.1. *Let M and N be nonzero R-modules, and let $n \geq 1$ be an integer. Assume:*

(i) *M is Tor-rigid.*
(ii) $\mathsf{CI\text{-}dim}(N) < \infty$.
(iii) *$M \otimes_R N$ satisfies $(\widetilde{S_n})$.*
(iv) *$\mathsf{Tor}_i^R(M,N)$ is torsion for all $i \gg 0$.*

Then $\mathsf{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, and N satisfies $(\widetilde{S_n})$. □

To prove Theorem 3.1, we will establish several lemmas.

Lemma 3.2. Let $0 \to N \xrightarrow{\mu} F \to N_1 \to 0$ be a short exact sequence of R-modules, where F is free and $\mathsf{Ext}_R^1(N_1, R) = 0$. Then it follows that $\mathsf{Ext}_R^1(\mathsf{Tr}N, M) \cong \mathsf{Tor}_1^R(N_1, M)$.

Proof. We consider the following commutative diagram, where the horizontal maps are the natural ones and $\mathsf{Hom}(\mu^*, M)$ is injective:

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\chi} & \mathsf{Hom}_R(N^*, M) \\ \downarrow{\scriptstyle \mu \otimes M} & & \downarrow{\scriptstyle \mathsf{Hom}(\mu^*,\, M)} \\ M \otimes_R F & \xrightarrow{\cong} & \mathsf{Hom}_R(F^*, M) \end{array}$$

Note that $\mathsf{Ext}_R^1(\mathsf{Tr}N, M) = \ker(\chi)$; see 2.2. Hence it follows from the above diagram that $\mathsf{Ext}_R^1(\mathsf{Tr}N, M) = \ker(\chi) \cong \ker(\mu \otimes M) = \mathsf{Tor}_1^R(N_1, M)$, as required. □

Lemma 3.3. Let M and N be R-modules with $\mathsf{CI\text{-}dim}_R(N) < \infty$. If $\mathsf{Tor}_i^R(M,N)$ is torsion for all $i \gg 0$, then $\mathsf{Tor}_i^R(M,N)$ and $\mathsf{Ext}_R^i(\mathsf{Tr}N, M)$ are torsion for each $i \geq 1$; cf., [7, A.2.].

Proof. Let $\mathfrak{p} \in \mathsf{Y}^0(R)$. Then, since $\mathsf{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ for all $i \gg 0$ and $\mathsf{CI\text{-}dim}_{R_\mathfrak{p}}(N_\mathfrak{p}) = 0$, we conclude that $\mathsf{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ for all $i \geq 1$ and also $\mathsf{Ext}_{R_\mathfrak{p}}^i(\mathsf{Tr}_{R_\mathfrak{p}} N_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $i \geq 1$; see 2.8 and [5, 4.9]. □

Lemma 3.4. Let M and N be R-modules such that $M \neq 0$ and M is Tor-rigid. If $n \geq 1$ is an integer and $\mathsf{Ext}_R^n(N,M) = 0$, then $\mathsf{Ext}_R^n(N,R) = 0$.

Proof. It follows from [2, 2.8(b)] that there is an exact sequence:

$$\mathsf{Tor}_2^R(\mathsf{Tr}\Omega^n N, M) \to \mathsf{Ext}_R^n(N,R) \otimes_R M \to \mathsf{Ext}_R^n(N,M) \to \mathsf{Tor}_1^R(\mathsf{Tr}\Omega^n N, M) \to 0.$$

As $\mathsf{Ext}_R^n(N,M) = 0$ and M is Tor-rigid, we have that $\mathsf{Tor}_2^R(\mathsf{Tr}\Omega^n N, M) = 0$. Thus we conclude $\mathsf{Ext}_R^n(N,R) \otimes_R M \cong \mathsf{Ext}_R^n(N,M) = 0$. This gives, since $M \neq 0$, that $\mathsf{Ext}_R^n(N,R) = 0$. □

We are now ready to give a proof for our main result:

Proof of Theorem 3.1. Note, to show N satisfies $(\widetilde{S_n})$, in view of 2.6 and Lemma 3.4, it suffices to prove $\mathsf{Ext}^i_R(\mathsf{Tr}N,M)=0$ for each $i=1,\ldots,n$. The vanishing of $\mathsf{Ext}^i_R(\mathsf{Tr}N,M)$, as well as that of $\mathsf{Tor}^R_i(M,N)$, is clear if $\mathsf{depth}(R)=0$; see Lemma 3.3. So we assume $\mathsf{depth}(R)\geq 1$.

It follows from 2.2 that there is an injection: $\mathsf{Ext}^1_R(\mathsf{Tr}N,M)\hookrightarrow M\otimes_R N$. It is easy to see, since $M\otimes_R N$ satisfies $(\widetilde{S_1})$, that $M\otimes_R N$ is torsion-free. On the other hand, $\mathsf{Ext}^1_R(\mathsf{Tr}N,M)$ is torsion; see Lemma 3.3. This establishes the theorem for the case where $n=1$, and also yield the vanishing of $\mathsf{Tor}^R_i(M,N)$ as we observe next: it follows from Lemma 3.4 that $\mathsf{Ext}^1_R(\mathsf{Tr}N,R)=0$, and hence we can consider the pushforward N_1 of N; see 2.6 and 2.3. Now Lemma 3.2 shows $\mathsf{Ext}^1_R(\mathsf{Tr}N,M)=0=\mathsf{Tor}^R_1(N_1,M)$. As M is Tor-rigid, we have $\mathsf{Tor}^R_i(M,N)=0$ for each $i\geq 1$.

Next we assume $n\geq 2$, and proceed by induction on n to show that N satisfies $(\widetilde{S_n})$. Suppose there is an integer t such that $t<n$ and $\mathsf{Ext}^i_R(\mathsf{Tr}N,M)=0$ for each $i=1,\ldots,t$. Our aim is to prove the vanishing of $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)$.

It follows from Lemma 3.4, that $\mathsf{Ext}^i_R(\mathsf{Tr}N,R)=0$ for each $i=1,\ldots,t$, i.e., N satisfies $(\widetilde{S_t})$. Therefore we can consider the pushforward sequences

$$0\to N_{i-1}\to F_i\to N_i\to 0, \tag{3.1.1}$$

where $N_0=N$, F_i is free and $\mathsf{Ext}^1_R(N_i,R)=0$ for each $i=1,\ldots,t$; see 2.3.

Note that, for each $i=1,\ldots,t$, we have:

$$\mathsf{Tor}^R_1(M,N_i)\cong \mathsf{Ext}^1_R(\mathsf{Tr}N_{i-1},M)\cong \mathsf{Ext}^i_R(\mathsf{Tr}N,M)=0. \tag{3.1.2}$$

Here, the first isomorphism in (3.1.2) is due to Lemma 3.2, while the second isomorphism follows since $\Omega^{i-1}\mathsf{Tr}N\cong \mathsf{Tr}N_{i-1}$ for $i=1,\ldots t$; see 2.3.

Now, in view of (3.1.2), tensoring the short exact sequences in (3.1.1) with M, we obtain the following short exact sequences for each $i=1,\ldots,t$:

$$0\to M\otimes_R N_{i-1}\to M\otimes_R F_i\to M\otimes_R N_i\to 0. \tag{3.1.3}$$

Recall our aim is to show that $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)=0$, and since $\Omega^t\mathsf{Tr}N\cong\mathsf{Tr}N_t$ (up to free summands), we have $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)\cong\mathsf{Ext}^1_R(\mathsf{Tr}N_t,M)$; see 2.3. So 2.2 yields an injection as:

$$\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)\hookrightarrow M\otimes_R N_t. \tag{3.1.4}$$

Next we assume $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)\neq 0$, pick $\mathfrak{q}\in\mathrm{Ass}(\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M))$, and seek a contradiction.

Suppose $\mathfrak{q}\in\mathsf{Y}^t(R)$. Then, since N satisfies $(\widetilde{S_t})$, we have $\mathsf{depth}_{R_\mathfrak{q}}(N_\mathfrak{q})\geq\mathsf{depth}(R_\mathfrak{q})$. This shows $\mathsf{CI\text{-}dim}_{R_\mathfrak{q}}(N_\mathfrak{q})=\mathsf{depth}(R_\mathfrak{q})-\mathsf{depth}_{R_\mathfrak{q}}(N_\mathfrak{q})=0$. Therefore, since $\mathsf{Tor}^R_i(M,N)=0$ for each $i\geq 1$, we deduce from 2.8 that $\mathsf{Ext}^i_R(\mathsf{Tr}N,M)_\mathfrak{q}=0$ for each $i\geq 1$. In particular $\mathfrak{q}\notin\mathsf{Y}^t(R)$, i.e., $\mathsf{depth}(R_\mathfrak{q})\geq t+1$, because of the fact that $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)_\mathfrak{q}\neq 0$.

Notice $\mathfrak{q}\in\mathrm{Supp}(M)\cap\mathrm{Supp}(N)$. Hence it follows from 2.7 that

$$\mathsf{depth}_{R_\mathfrak{q}}(M_\mathfrak{q})=\big(\mathsf{depth}(R_\mathfrak{q})-\mathsf{depth}_{R_\mathfrak{q}}(N_\mathfrak{q})\big)+\mathsf{depth}_{R_\mathfrak{q}}(M_\mathfrak{q}\otimes_{R_\mathfrak{q}}N_\mathfrak{q})\geq t+1. \tag{3.1.5}$$

The inequality in (3.1.5) are due to the following facts: $t+1\leq n$ so that $M\otimes_R N$ satisfies $(\widetilde{S_{t+1}})$, $\mathsf{depth}(R_\mathfrak{q})\geq t+1$, and $\mathsf{CI\text{-}dim}_{R_\mathfrak{q}}(N_\mathfrak{q})=\mathsf{depth}(R_\mathfrak{q})-\mathsf{depth}_{R_\mathfrak{q}}(N_\mathfrak{q})\geq 0$.

Recall that $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)_\mathfrak{q}$ is a nonzero module of depth zero. Hence, we see, by revisiting (3.1.4), that $\mathsf{depth}_{R_\mathfrak{q}}(M_\mathfrak{q}\otimes_{R_\mathfrak{q}}(N_t)_\mathfrak{q})=0$. However, by localizing (3.1.3) at $\mathfrak{q}$ and using depth lemma, along with (3.1.5), we have $\mathsf{depth}_{R_\mathfrak{q}}(M_\mathfrak{q}\otimes_{R_\mathfrak{q}}N_\mathfrak{q})=t$; this is a contradiction since $M\otimes_R N$ satisfies $(\widetilde{S_{t+1}})$ and so $\mathsf{depth}_{R_\mathfrak{q}}(M_\mathfrak{q}\otimes_{R_\mathfrak{q}}N_\mathfrak{q})\geq t+1$. Consequently, $\mathsf{Ext}^{t+1}_R(\mathsf{Tr}N,M)$ must vanish, and this completes the proof of the theorem. □

4. Proof of Theorem 1.2 and further remarks

Definition 4.1. ([15]) The *Hochster-Huneke graph* $G(R)$ is defined as follows:

- The set of *vertices* equals $\mathrm{Min}(R)$, i.e., vertices are the minimal prime ideals of R.
- There is an *edge* between two vertices $\mathfrak{p}$ and $\mathfrak{q}$ of $G(R) \Longleftrightarrow \mathsf{height}(\mathfrak{p}+\mathfrak{q}) \leq 1$.

Remark 4.2. ([15]) The following hold for the graph $G(R)$:

(i) Given two vertices $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of $G(R)$, there is an edge between $\mathfrak{p}_1$ and $\mathfrak{p}_2$ if and only if $\mathfrak{p}_1+\mathfrak{p}_2$ is contained in some height-one prime ideal $\mathfrak{q}$.

(ii) $G(R)$ is *connected* if and only if given two vertices $\mathfrak{p}$ and $\mathfrak{p}'$ of $G(R)$, there are minimal prime ideals $\{\mathfrak{p}_0,\mathfrak{p}_1,\dots,\mathfrak{p}_r\}$ of R, and height-one prime ideals $\{\mathfrak{q}_1,\mathfrak{q}_2,\dots,\mathfrak{q}_r\}$ of R, where $\mathfrak{p}=\mathfrak{p}_0$, $\mathfrak{p}'=\mathfrak{p}_r$ and $\mathfrak{p}_i,\mathfrak{p}_{i+1} \subseteq \mathfrak{q}_{i+1}$ for each $i=0,1,\dots,r-1$. □

The first part of the next proposition is proved in [15, 3.6] for complete local rings. Here, for the convenience of the reader, we go over its proof since we do not assume R is complete.

Proposition 4.3. *Assume R satisfies (S_2), e.g., R is Cohen-Macaulay. Then the following hold:*

(i) *$G(R)$ is connected.*

(ii) *If N is an R-module such that $N_{\mathfrak{p}}$ is free for each $\mathfrak{p} \in Y^1(R)$, then N has rank.*

Proof. (i) We assume $G(R)$ is not connected, and seek a contradiction.

Notice, since $G(R)$ is disconnected, there is a nontrivial partition of the set of all minimal prime ideals of R as $\mathrm{Min}(R)=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}\sqcup\{\mathfrak{q}_1,\cdots,\mathfrak{q}_s\}$, where $\mathsf{height}(\mathfrak{p}_i+\mathfrak{q}_j)\geq 2$ for each i and j. Letting $I=\bigcap_{i=1}^r \mathfrak{p}_i$ and $J=\bigcap_{j=1}^s \mathfrak{q}_j$, we get two non-nilpotent ideals I and J such that IJ is nilpotent. Moreover it follows that $\mathsf{height}(I+J)\geq 2$ since

$$V(I+J)=V(I)\cap V(J)=\left[\bigcup_{i=1}^{r} V(\mathfrak{p}_i)\right]\bigcap\left[\bigcup_{j=1}^{s} V(\mathfrak{q}_j)\right]=\bigcup_{i,j} V(\mathfrak{p}_i+\mathfrak{q}_j) \text{ and } \mathsf{height}(\mathfrak{p}_i+\mathfrak{q}_j)\geq 2.$$

By replacing the ideals I and J with their appropriate powers, we may assume $IJ=0$.

Since R satisfies (S_2) and $\mathsf{height}(I+J)\geq 2$, there is an R-regular sequence $\{u+v,u'+v'\}$ in $I+J$, where $u,u'\in I$ and $v,v'\in J$. In view of the fact $v'(u+v)-v(u'+v')=v'u-vu'\in IJ=0$, we conclude that there is an element $a\in R$ such that $v=a(u+v)$. Similarly, we deduce that $u=b(u+v)$ for some $b\in R$. Therefore we have $u+v=(a+b)(u+v)$, and hence $a+b$ is unit in R. This implies that either a or b is unit in R. We assume, without loss of generality, that a is unit. Then u is R-regular, and the equality $uJ=0$ shows that $J=0$, which is a contradiction. Consequently, $G(R)$ is not connected.

(ii) Note, as R satisfies (S_2), each associated prime of R is minimal, and $\mathfrak{p}\in Y^1(R)$ if and only if $\mathsf{height}(\mathfrak{p})\leq 1$. Moreover, by part (i), we know $G(R)$ is connected.

Let $\mathfrak{p}$ and $\mathfrak{p}'$ be two minimal prime ideals of R. Then we know there are minimal prime ideals $\{\mathfrak{p}_0,\mathfrak{p}_1,\dots,\mathfrak{p}_r\}$ of R, and height-one prime ideals $\{\mathfrak{q}_1,\mathfrak{q}_2,\dots,\mathfrak{q}_r\}$ of R, where $\mathfrak{p}=\mathfrak{p}_0$, $\mathfrak{p}'=\mathfrak{p}_r$ and $\mathfrak{p}_i,\mathfrak{p}_{i+1}\subseteq\mathfrak{q}_{i+1}$ for each $i=0,1,\dots,r-1$.

By assumption, for each $i=0,1,\dots,r-1$, we know that the modules $M_{\mathfrak{p}_i}$, $M_{\mathfrak{p}_{i+1}}$ and $M_{\mathfrak{q}_{i+1}}$ are free. Moreover, as $(M_{\mathfrak{q}_{i+1}})_{\mathfrak{p}_i R_{\mathfrak{q}_{i+1}}}\cong M_{\mathfrak{p}_i}$, for each $i=0,1,\dots,r-1$, we deduce:

$$\mathsf{rank}_{R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})=\mathsf{rank}_{R_{\mathfrak{q}_{i+1}}}(M_{\mathfrak{q}_{i+1}})=\mathsf{rank}_{R_{\mathfrak{p}_{i+1}}}(M_{\mathfrak{p}_{i+1}}).$$

This shows that $\mathsf{rank}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=\mathsf{rank}_{R_{\mathfrak{p}'}}(M_{\mathfrak{p}'})$, as required. □

We can strengthen the conclusion of Theorem 3.1, and show that both modules in question satisfy $(\widetilde{S_n})$ in case local freeness hypothesis on $\mathsf{Y}^1(R)$ is included in our assumptions.

Corollary 4.4. *Assume R satisfies (S_2), n is a positive integer, and M and N are nonzero R-modules. Assume further:*

(i) *M is Tor-rigid.*
(ii) $\mathsf{CI\text{-}dim}(N) < \infty$.
(iii) *$M \otimes_R N$ satisfies $(\widetilde{S_n})$.*
(iv) $\mathsf{pd}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \infty$ *for each* $\mathfrak{p} \in \mathsf{Y}^1(R)$.

Then $\mathsf{Tor}_i^R(M,N) = 0$ *for all* $i \geq 1$*, and both M and N satisfy* $(\widetilde{S_n})$.

Proof. It follows from Theorem 3.1 that $\mathsf{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, and N satisfies $(\widetilde{S_n})$. Note that, both $M \otimes_R N$ and N satisfy (S_1). Hence $N_{\mathfrak{p}}$ is free for each $\mathfrak{p} \in \mathsf{Y}^1(R)$. In particular, N has rank due to Proposition 4.3(ii). Therefore, since $M \otimes_R N$ is torsion-free, we conclude that $\operatorname{Supp}(N) = \operatorname{Spec}(R)$. Now the depth formula shows M satisfies $(\widetilde{S_n})$; see 2.7 and [8, 1.3]. □

Now we can prove Theorem 1.2, the result advertised in the introduction:

Proof of Theorem 1.2. The result is an immediate consequence of Corollary 4.4 since M is Tor-rigid by a result of Lichtenbaum; see 2.1(i). □

Next we recall an example given in [10] concerning the Second Rigidity Theorem; see Theorem 1.1. The presentation we provide for $M \otimes_R N$ in Example 4.5 has not been given in [10] and appears to be new; here we compute it by using [14, 21].

Example 4.5. ([10]) Let $R = \mathbb{C}[\![x,y,z,w]\!]/(xy)$, $M = \mathsf{Tr}(R/\mathfrak{p})$, where $\mathfrak{p} = (y,z,w) \in \operatorname{Spec}(R)$, and let $N = R/(x)$. Then M is not reflexive, but since $\mathsf{pd}(M) < \infty$, we have that N is reflexive by Theorem 1.1. Moreover, $M \otimes_R N$ is reflexive since it is the second syzygy of the cokernel of the rightmost matrix in the following exact sequence:

$$R^{\oplus 4} \xrightarrow{\begin{pmatrix} x&0&0&w\\ 0&x&0&y\\ 0&0&x&z\end{pmatrix}} R^{\oplus 3} \xrightarrow{\begin{pmatrix}0&yz&-y^2\\ -yz&0&yz\\ y^2&-yw&0\end{pmatrix}} R^{\oplus 3} \xrightarrow{\begin{pmatrix}x&0&0\\ 0&x&0\\ 0&0&x\\ w&y&z\end{pmatrix}} R^{\oplus 4}$$

$$\begin{array}{ccccc} R^{\oplus 3} & & & & R^{\oplus 3}\\ & \searrow & & \nearrow & \\ & & M\otimes_R N & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0\end{array}$$

□

Next we point out that the conclusion of Theorem 1.2 is sharp:

Remark 4.6. In Example 4.5, it follows, as $\mathsf{pd}(M) < \infty$, that $\mathsf{Tor}_i^R(M,N)_{\mathfrak{p}} = 0$ for all $i \gg 0$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$, but M is not reflexive. In other words, the torsion hypothesis (iv) of Theorem 3.1 is not enough to obtain the conclusion of Theorem 1.2, in general.

We can easily see that there is a height-one prime ideal $\mathfrak{q}$ of R in Example 4.5 such that $\mathsf{pd}_{R_\mathfrak{q}}(N_\mathfrak{q}) = \infty$. For that note the minimal free resolution of N is given as:

$$\dots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow N \longrightarrow 0.$$

Localizing this resolution at the height-one prime ideal $\mathfrak{q} = (x,y)$ of R, we obtain the minimal free resolution of $N_\mathfrak{q}$ over $R_\mathfrak{q}$:

$$\dots \xrightarrow{y} R_\mathfrak{q} \xrightarrow{x} R_\mathfrak{q} \xrightarrow{y} R_\mathfrak{q} \xrightarrow{x} R_\mathfrak{q} \longrightarrow N_\mathfrak{q} \longrightarrow 0.$$

This clearly shows that $\mathsf{pd}_{R_\mathfrak{q}}(N_\mathfrak{q}) = \infty$. □

An R-module M is said to be 2-*Tor-rigid* provided, whenever $\mathsf{Tor}_1^R(M,N) = 0 = \mathsf{Tor}_2^R(M,N)$ for some R-module N, we have $\mathsf{Tor}_3^R(M,N) = 0$. We finish this section by noting that the conclusion of Theorem 3.1 may fail if the module M is 2-Tor-rigid instead of Tor-rigid:

Example 4.7. Let $R = \mathbb{C}[\![x,y]\!]/(xy)$, $M = R/(x)$ and $N = R/(x^2)$. Note that each R-module is 2-Tor-rigid [22, 1.9]. Note also that $M \otimes_R N \cong M$ and hence $M \otimes_R N$ satisfies $(\widetilde{S_v})$ for each $v \geq 0$. Also, since R is reduced, $\mathsf{Tor}_i^R(M,N)$ is torsion for each $i \geq 1$. However it is easy to see that N does not satisfy $(\widetilde{S_1})$, $\mathsf{Tor}_1^R(M,N) \neq 0$, and M is not Tor-rigid; see [17, page 164]. □

It is worth noting that we do not know an example similar to Example 4.7 when $n \geq 2$. More precisely, we ask (cf. Example 4.5):

Question 4.8. Let R be a hypersurface ring, and let M and N be nonzero R-modules. Assume $\mathsf{Tor}_i^R(M,N)$ is torsion for all $i \gg 0$. If $M \otimes_R N$ is reflexive, then must M *or* N be reflexive? □

Notice, if the ring R in Question 4.8 is a domain (e.g., an isolated singularity of dimension at least two), then it follows from 2.7 that both M and N are reflexive; see [8, 1.3].

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