DEPTH FORMULA FOR MODULES OF FINITE
REDUCING PROJECTIVE DIMENSION

OLGUR CELIKBAS, TOSHINORI KOBAYASHI, BRIAN LAVERTY, AND HIROKI MATSUI

ABSTRACT. We prove that the depth formula holds for two finitely generated Tor-independent modules over Cohen-Macaulay local rings if one of the modules considered has finite reducing projective dimension (for example, if it has finite projective dimension, or the ring is a complete intersection). This generalizes a result of Bergh-Jorgensen which shows that the depth formula holds for two finitely generated Tor-independent modules over Cohen-Macaulay local rings if one of the modules considered has reducible complexity and certain additional conditions hold.

Each module that has reducible complexity also has finite complexity and finite reducing projective dimension, but not necessarily vice versa. So a new advantage we have is that, unlike modules of reducible complexity, Betti numbers of modules of finite reducing projective dimension can grow exponentially.

1. INTRODUCTION

Throughout R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and all R-modules are assumed to be finitely generated.

In his beautiful paper [4] Auslander proved that, if M and N are R-modules such that $\operatorname{pd}_R(M) < \infty$, and M and N are Tor-independent, that is, if $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, then the following holds:

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(M \otimes_R N).$$

Huneke-Wiegand dubbed this depth equality the *depth formula*, and proved that the depth formula holds if R is a complete intersection ring, and M and N are Tor-independent R-modules, regardless of whether M or N has finite projective dimension; see [27, 2.5]. Subsequently, consequences of the depth formula, and different conditions that force this formula to hold were studied in several papers; see, for example, [3, 14, 18, 19, 20, 21, 25, 28]. In this paper we are concerned with results of Bergh-Jorgensen [12] who studied the depth formula over Cohen-Macaulay rings for modules that have reducible complexity; see also Bergh [11] and Sadeghi [36] for similar results over local rings that are not necessarily Cohen-Macaulay. The results of Bergh-Jorgensen in [12] concerning the depth formula can be summarized as follows; see also 4.1 and Remark 4.3.

Theorem 1.1 (Bergh-Jorgensen; see [12, 3.3, 3.4, 3.5, 3.6]). *Let R be a Cohen-Macaulay local ring and let M and N be Tor-independent R-modules. Assume at least one of the following holds:*

(i) $\operatorname{depth}_R(M) < \operatorname{depth}(R)$.
(ii) $\operatorname{G-dim}_R(N) < \infty$ *(for example, R is Gorenstein).*
(iii) *N has reducible complexity (for example, R is a complete intersection ring).*

If M has reducible complexity, then the depth formula holds for M and N. □

In this paper we generalize Theorem 1.1 and prove the following in Theorem 3.4:

Theorem 1.2. *Let R be a Cohen-Macaulay local ring and let M and N be Tor-independent R-modules. If M has finite reducing projective dimension (for example, if M has reducible complexity), then the depth formula holds for M and N.* □

We recall the definitions of reducible complexity and reducing projective dimension in the next section; see 2.2 and 2.3. Here let us note that finite reducing homological dimension generalizes finite homological dimension. The definition of reducing projective dimension [1] was motivated by that of reducible complexity [11], but these two definitions are different in nature: The latter deals with modules that have finite complexity, namely, modules that have polynomial growth on their Betti numbers. It is worth noting that, for an R-module M, we have the following (non)implications; see 2.1, 2.2, and 2.3.

$$\mathrm{pd}_R(M) < \infty \; \overrightarrow{\underleftarrow{}} \; \mathrm{CI\text{-}dim}_R(M) < \infty \; \overrightarrow{\underleftarrow{}} \; M \text{ has reducible complexity} \; \overrightarrow{\underleftarrow{}} \; \mathrm{red\text{-}pd}_R(M) < \infty$$

As noted above, each module that has reducible complexity also has (by definition) finite reducing projective dimension, but the converse of this fact is not true in general. For example, if R is a Cohen-Macaulay non-Gorenstein local ring such that R has minimal multiplicity and $|k| = \infty$ (one can pick $R = \mathbb{C}[\![t^3, t^4, t^5]\!]$) and $M = \Omega_R^n(k)$ for some $n \geq 0$, then M has finite reducing projective dimension, but M does not have reducible complexity; see [16, 1.2]. There are many similar examples in the literature; to name a few, see Examples 2.9, 2.10, and 2.11.

Recently, several classical results for modules with finite homological dimensions have been generalized in terms of reducing homological dimensions. For example, a local ring R is Gorenstein if and only if each R-module has finite reducing Gorenstein dimension; see 2.8(iii). Theorem 1.2 is another such contribution to the literature and it is a special case of a more general result, namely Theorem 3.4; see also Proposition 3.2. Theorem 3.4, our main result, establishes the derived version of the depth formula [23] over Cohen-Macaulay local rings provided that Tor modules eventually vanish and one of the modules considered has finite reducing projective dimension. The proof of Theorem 3.4 relies upon the usage of some derived category tools; see, for example, 2.15 and 2.16.

In section 2, we recall the necessary definitions and provide several examples. The proof of Theorem 1.2 is given in section 3, while section 5 contains the proofs of the auxiliary results used in proving the theorem. Section 4 is devoted to some applications of our results; see, for example, Corollary 4.2 and Proposition 4.6. One such application considers a beautiful formula of Jorgensen [29], which extends the classical Auslander-Buchsbaum formula; see 4.14. Over Cohen-Macaulay rings, we improve Jorgensen's formula and prove the following; see Theorem 4.15.

Theorem 1.3. *Let R be a Cohen-Macaulay local ring and let M and N be nonzero R-modules. If* $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i \gg 0$ *and* $\mathrm{red\text{-}CI\text{-}dim}_R(M) < \infty$, *then*

$$\sup\{i : \mathrm{Tor}_i^R(M, N) \neq 0\} = \sup\Big\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\Big\}.$$

2. Definitions, preliminary results, and examples

In this section we record some definitions, and discuss several examples and preliminary results that are needed in the subsequent sections.

2.1 (**Gorenstein and complete intersection dimensions** [7, 10]). An R-module M is said to be *totally reflexive* provided that the natural map $M \to M^{**}$, where $M^* = \mathrm{Hom}_R(M, R)$, is bijective and $\mathrm{Ext}_R^i(M, R) = 0 = \mathrm{Ext}_R^i(M^*, R)$ for all $i \geq 1$. The infimum of n for which there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$, such that each X_i is totally reflexive, is called the *Gorenstein dimension* of M. If M has Gorenstein dimension n, we write $\mathrm{G\text{-}dim}_R(M) = n$. Therefore, M is totally reflexive if and only if $\mathrm{G\text{-}dim}_R(M) \leq 0$, where it follows by convention that $\mathrm{G\text{-}dim}_R(0) = -\infty$.

A diagram of local ring maps $R \to R' \twoheadleftarrow S$ is called a *quasi-deformation* if $R \to R'$ is flat and the kernel of the surjection $R' \twoheadleftarrow S$ is generated by a regular sequence on S. The *complete intersection*

dimension of M is defined as follows:

$$\text{CI-dim}_R(M) = \inf\{\text{pd}_S(M \otimes_R R') - \text{pd}_S(R') : R \to R' \leftarrow S \text{ is a quasi-deformation}\}.$$

Note that, if R is a complete intersection ring, then each R-module has finite complete intersection dimension [10, 1.3]. Moreover, the inequalities $\text{G-dim}_R(M) \leq \text{CI-dim}_R(M) \leq \text{pd}_R(M)$ hold in general; if one of these dimensions is finite, then it is equal to the one to its left; see [10, 1.4].

We set $\text{pd}_R(0) = \text{G-dim}_R(0) = \text{CI-dim}_R(0) = -\infty$ and also $\text{depth}_R(0) = \infty$. $\qquad \square$

The reducing projective dimension definition [1] originated from the reducible complexity definition, a notion introduced by Bergh; see [11]. Hence we first recall the definition of reducible complexity. In the following $\Omega_R^i M$ denotes the i-th syzygy of a given R-module M in its minimal free resolution.

2.2 (Complexity and reducible complexity [9, 11])**.** Let M be an R-module. The *complexity* of M [9] is:

$$\text{cx}_R(M) = \inf\{r \in \mathbb{N}_0 \mid \exists A \in \mathbb{R} \text{ such that } \dim_k\left(\text{Tor}_n^R(M,k)\right) \leq A \cdot n^{r-1} \text{ for all } n \gg 0\}.$$

Note that $\text{cx}_R(M) = 0$ if and only if $\text{pd}_R(M) < \infty$, and $\text{cx}_R(M) \leq 1$ if and only if M has bounded Betti numbers. In general $\text{cx}_R(M)$ may be infinite; in fact, R is a complete intersection if and only if $\text{cx}_R(k) < \infty$; see [10, 1.3]. Moreover, if $\text{CI-dim}_R(M) < \infty$ (for example, if R is a complete intersection ring), then $\text{cx}_R(M) < \infty$; see [10, 5.3].

The module M is said to have *reducible complexity* [11] if $\text{cx}_R(M) = 0$, or if $0 < \text{cx}_R(M) = r < \infty$ and there are short exact sequences of R-modules

$$\{0 \to K_{i-1} \to K_i \to \Omega_R^{n_i} K_{i-1} \to 0\}_{i=0}^r,$$

with $n_i \geq 0$, where $K_0 = M$ and $\text{cx}_R(K_{i+1}) = \text{cx}_R(K_i) - 1$ for all $i = 0, \ldots, r$ (so that $\text{pd}_R(K_r) < \infty$).

There are many examples of modules that have reducible complexity. For example, periodic modules over arbitrary local rings, and modules over complete intersection rings (or more generally, modules over arbitrary local rings that have finite complete intersection dimension) have reducible complexity; see [10, 11] for the details.

2.3. (Reducing invariants [1, 2]) Let M be an R-module and let $\mathbb{I}_R$ denote an invariant of R-modules, that is, $\mathbb{I}_R$ denotes a function from the set of isomorphism classes of R-modules to the set $\mathbb{Z} \cup \{\pm\infty\}$. Classical examples of $\mathbb{I}_R$ are the projective dimension pd_R, the Gorenstein dimension G-dim_R, and the complete intersection dimension CI-dim_R.

The *reducing invariant* red-$\mathbb{I}_R(M)$ of M is zero if $\mathbb{I}_R(M) < \infty$. Hence, if $\mathbb{I}_R \in \{\text{pd}_R, \text{G-dim}_R, \text{CI-dim}_R\}$, then red-$\mathbb{I}_R(0) = 0$ since $\mathbb{I}_R(0) = -\infty$; see [5]. If $\mathbb{I}_R(M) = \infty$, we write red-$\mathbb{I}_R(M) < \infty$ provided that there exist integers $r \geq 1$, $a_i \geq 1$, $b_i \geq 1$, $n_i \geq 0$, and short exact sequences of R-modules of the form

$$(2.3.1) \qquad\qquad 0 \to K_{i-1}^{\oplus a_i} \to K_i \to \Omega_R^{n_i} K_{i-1}^{\oplus b_i} \to 0,$$

for each $i = 1, \ldots, r$, where $K_0 = M$ and $\mathbb{I}_R(K_r) < \infty$. If a sequence as in (2.3.1) exists, then we call $\{K_0, \ldots, K_r\}$ a *reducing $\mathbb{I}$-sequence* of M; see [1, 2.1] and [2, 2.5].

If $\mathbb{I}_R(M) = \infty$, we define:

$$\text{red-}\mathbb{I}_R(M) = \inf\{r \in \mathbb{N} : \text{there is a reducing } \mathbb{I}\text{-sequence } \{K_0, \ldots, K_r\} \text{ of } M\}.$$

It follows that $0 \leq \text{red-}\mathbb{I}_R(M) \leq \infty$. $\qquad \square$

Remark 2.4. In the definition of reducible complexity and reducing invariants, minimal syzygy modules are used. However, if $\mathbb{I}_R \in \{\text{pd}_R, \text{G-dim}_R, \text{CI-dim}_R\}$, then we may use syzygies that are not necessarily minimal to define red-$\mathbb{I}_R$. This can be seen as follows:

If $0 \to K \to K_1 \to \Omega_R^n K \oplus F \to 0$ is a short exact sequence of R-modules, where F is a free module, then we obtain an induced exact sequence $0 \to K \to K_1' \to \Omega_R^n K \to 0$, where $K_1 \cong K_1' \oplus F$ and red-$\mathbb{I}_R(K_1) = \text{red-}\mathbb{I}_R(K_1')$; see [16, 3.8 and the proof of the claim in the proof of 4.2]. Hence, if M is an R-module, then the value of red-$\mathbb{I}_R(M)$ does not change whether or not we use minimal syzygies to define red-$\mathbb{I}_R$. $\qquad \square$

In passing we make use of Remark 2.4 and note:

Lemma 2.5. *Let M be an R-module and let $\mathbb{I}_R \in \{\mathrm{pd}_R, \mathrm{G\text{-}dim}_R, \mathrm{CI\text{-}dim}_R\}$. Then* $\mathrm{red\text{-}}\mathbb{I}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{red\text{-}}\mathbb{I}_R(M)$ *for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Proof. If $\mathfrak{p} \in \mathrm{Spec}(R)$, then the localization of a reducing $\mathbb{I}$-sequence of M over R at $\mathfrak{p}$ is a reducing $\mathbb{I}$-sequence of $M_\mathfrak{p}$ over $R_\mathfrak{p}$; see 2.3 and Remark 2.4. So $\mathrm{red\text{-}}\mathbb{I}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{red\text{-}}\mathbb{I}_R(M)$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.   □

Next are some further results that are needed later in the sequel.

2.6. The following properties are due to the definition of reducing invariants; see 2.3 and also [16, 2.2]:
- (i) If $1 \leq \mathrm{red\text{-}}\mathbb{I}_R(M) < \infty$, then there is an exact sequence $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers, and K is an R-module such that $\mathrm{red\text{-}}\mathbb{I}_R(K) = \mathrm{red\text{-}}\mathbb{I}_R(M) - 1$.
- (ii) Conversely, if $0 \to M^{\oplus a} \to K \to \Omega_R^n N^{\oplus b} \to 0$ is an exact sequence of R-modules, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers, then $\mathrm{red\text{-}}\mathbb{I}_R(M) \leq \mathrm{red\text{-}}\mathbb{I}_R(K) + 1$.

2.7. Let M be an R-module.
- (i) $\mathrm{red\text{-}CI\text{-}dim}_R(M) < \infty$ if and only if $\mathrm{red\text{-}pd}_R(M) < \infty$; see [15, 2.7].
- (ii) If $x \in \mathfrak{m}$ is a non zero-divisor on R and M, then $\mathrm{red\text{-}pd}_{R/xR}(M/xM) \leq \mathrm{red\text{-}pd}_R(M)$; see [16, 3.4].

Remark 2.8.
- (i) The definition of the reducing dimension we use in this paper is taken from [2], even though this notion was originally defined in [1]. The difference between these two definitions is that [2] only requires the integers n_i in Definition 2.3 to be nonnegative, while [1] requires these numbers to be positive. Hence, if a module has finite reducing invariant with respect to the definition given in [1], then it has also finite reducing invariant with respect to the definition we use in this paper.
- (ii) Given an integer $c \geq 1$, one can always find a ring R and an R-module M with $\mathrm{red\text{-}pd}_R(M) = c < \infty$ and $\mathrm{pd}_R(M) = \infty$. More precisely, if R is a singular complete intersection ring of codimension c (for example, $R = k[[x_1, \ldots, x_c, y_1, \ldots, y_c]]/(x_1 y_1, \ldots, x_c y_c)$ for some field k) and $M = \Omega_R^i(k)$ for an integer $i \geq 0$, then $\mathrm{pd}_R(M) = \infty$ and $\mathrm{red\text{-}pd}_R(M) = \mathrm{cx}_R(M) = c$; see [16, 2.8].
- (iii) Several characterizations of local rings in terms of reducing dimensions were obtained in [15]. For example, it follows that a local ring R is Gorenstein if and only if each R-module has finite reducing Gorenstein dimension; see [15, 3.12]. Similarly there are various questions that remain open about these invariants. For example, it is not known whether or not the residue field of a local ring always has finite reducing projective dimension; see [16] for the details.   □

In general, a module having reducible complexity also has finite reducing projective dimension; see 2.2 and 2.3. On the other hand, modules that have finite reducing projective dimension – which have infinite complexity – are abundant in the literature. In other words, reducing projective dimension is a finer invariant than reducible complexity. Next we record several examples which illustrate this fact.

Example 2.9 ([1, 2.3 and 2.4]). Let $R = k[[x, y]]/(x^2, xy, y^2)$. The beginning of the minimal free resolution of k is:

$$\cdots \longrightarrow R^{\oplus 4} \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow k \longrightarrow 0.$$

Notice R is not Gorenstein so that we have $\mathrm{cx}_R(k) = \mathrm{CI\text{-}dim}_R(k) = \mathrm{G\text{-}dim}_R(k) = \mathrm{pd}_R(k) = \infty$; see 2.1 and 2.2. As $\mathfrak{m} \cong k^{\oplus 2}$, it follows that $\Omega_R^2 k \cong k^{\oplus 4}$. So, there is an exact sequence $0 \to k^{\oplus 4} \to R^{\oplus 2} \to \Omega_R k \to 0$. Therefore $\{k, R^{\oplus 2}\}$ is a reducing pd-sequence of k. We see that $\mathrm{red\text{-}pd}_R(k) = \mathrm{red\text{-}G\text{-}dim}_R(k) = 1 < \infty$.

This construction can be generalized as long as $\mathfrak{m}^2 = 0$. In that case there is an exact sequence $0 \to k^{\oplus e^2} \to R^{\oplus e} \to \Omega_R k \to 0$, where e is the embedding dimension of R. This fact implies that $\{k, R^{\oplus e}\}$ is a reducing pd-sequence (and hence reducing G-dim) sequence of k. If R is not Gorenstein, then $\mathrm{red\text{-}pd}_R(k) = 1 = \mathrm{red\text{-}G\text{-}dim}_R(k)$ and $\mathrm{pd}_R(k) = \infty = \mathrm{G\text{-}dim}_R(k)$.

Example 2.10. ([1, 2.7]) Let $R = k[\![x^3, x^2y, xy^2, y^3]\!]$, the 3rd Veronese subring of the formal power series ring $k[\![x, y]\!]$, and consider the R-module $M = (x^2, xy, y^2)$. It follows that G-$\dim_R(M) = \infty$, and there exists a short exact sequence $0 \to M^{\oplus 4} \to F \to \Omega_R M \to 0$ for some free R-module F. Therefore we have that red-G-$\dim_R(M) = 1$.

Example 2.11. ([16, 2.12]) Let R be a Cohen-Macaulay local ring which is not regular. If I is an Ulrich ideal of R which is not a parameter ideal, then $\mathrm{pd}_R(R/I) = \infty$ and red-$\mathrm{pd}_R(R/I) = 1$.

If R has minimal multiplicity and $|k| = \infty$, then $\mathfrak{m}$ is an Ulrich ideal which is not a parameter ideal so that $\mathrm{pd}_R(k) = \infty$ and red-$\mathrm{pd}_R(k) = 1$. Some specific examples can be given as follows:

(i) If $R = k[\![x, y, z]\!]/(x^3 - y^2, z^2 - x^2y)$ and $I = (x, y)$. Then R is Cohen-Macaulay but not regular, and I is an Ulrich ideal of R which is not a parameter ideal. So $\mathrm{pd}_R(R/I) = \infty$ and red-$\mathrm{pd}_R(R/I) = 1$.

(ii) Let $R = \mathbb{C}[\![x, y]\!]/(x, y)^2$, or $R = \mathbb{C}[\![t^3, t^4, t^5]\!]$, or $R = \mathbb{C}[\![t^4, t^5, t^6, t^7]\!]$. Then R is a non-regular Cohen-Macaulay local ring with minimal multiplicity. It follows that $\mathrm{pd}_R(k) = \infty$ and red-$\mathrm{pd}_R(k) = 1$. $\quad\square$

2.12. Given R-modules M and N, we set $\mathrm{q}^R(M, N) = \sup\{i : \mathrm{Tor}_i^R(M, N) \neq 0\}$. Note that, if M and N are nonzero R-modules, then $\mathrm{q}^R(M, N) = 0$ if and only if M and N are Tor-independent.

We present several facts about complexes in 2.13 which are needed for our arguments. For example, a useful result of Iyengar [28, 2.3] recorded in 2.13(iii) allows us to avoid using spectral sequences in the proof of Proposition 3.2.

2.13. An R-complex X is a complex of (finitely generated) R-modules which is indexed homologically. Set $K = K(x_1, \ldots, x_e; R)$, the Koszul complex on a generating set $\{x_1, \ldots, x_e\}$ of $\mathfrak{m}$; see [28, 5.3(1)].

(i) We say X is homologically bounded if $\sup(X) - \inf(X) < \infty$, where $\inf(X) = \inf\{i \mid \mathrm{H}_i(X) \neq 0\}$ and $\sup(X) = \sup\{i \mid \mathrm{H}_i(X) \neq 0\}$. Also, the R-complex $X[1]$ is defined by $X[1]_n = X_{n-1}$ and $\partial_n^{X[1]} = -\partial_{n-1}^X$.

(ii) The depth of X is defined as $\mathrm{depth}_R(X) = -\sup(\mathrm{RHom}_R(k, X))$; see [28, 6.1]. It follows from [24, 2.1, 2.3] that $\mathrm{depth}_R(X)$ also equals $e - \sup(K \otimes_R X)$.

If $X \not\simeq 0$ and X is homologically bounded, then $\mathrm{depth}_R(X) \leq \dim(R) - \sup(X)$; see [22, 3.17]. Hence, if $X = M \otimes_R^{\mathbb{L}} N$ for some nonzero R-modules M and N such that $\mathrm{q}^R(M, N) < \infty$, then $\mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N) \leq \dim(R)$.

(iii) It follows that $\sup(X) \leq \sup(K \otimes_R X) \leq \sup(X) + e$ and $-\sup(X) \leq \mathrm{depth}_R(X) \leq e - \sup(X)$; see [24, 1.3]. Thus, $\mathrm{depth}_R(X) = -\infty$ if and only if $\sup(X) = \infty$; see also [28, Section 2].

Moreover, if $\sup(X) < \infty$ and $\mathrm{depth}_R(\mathrm{H}_s(X)) \leq 1$ with $s = \sup(X)$, then [28, 2.3] implies that $\mathrm{depth}_R(X) = \mathrm{depth}_R(\mathrm{H}_s(X)) - s$.

(iv) Let $X \to Y \to Z \to X[1]$ be an exact triangle, where X, Y, and Z are R-complexes such that $\sup(X) < \infty$ and $\sup(Z) < \infty$ (so that $\sup(Y) < \infty$). Then the *derived depth lemma* yields the following; see, for example, [13, 1.2.9].

  (a) $\mathrm{depth}_R(X) \geq \min\{\mathrm{depth}_R(Y), \mathrm{depth}_R(Z) + 1\}$.
  (b) $\mathrm{depth}_R(Y) \geq \min\{\mathrm{depth}_R(X), \mathrm{depth}_R(Z)\}$.
  (c) $\mathrm{depth}_R(Z) \geq \min\{\mathrm{depth}_R(Y), \mathrm{depth}_R(X) - 1\}$.
  (d) If $\mathrm{depth}_R(Z) \geq \mathrm{depth}_R(X)$, then $\mathrm{depth}_R(Y) = \mathrm{depth}_R(X)$.

(v) We say that the *derived depth formula* [20] holds for given R-modules M and N provided that

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N).$$

(vi) Let M and N be R-modules. If $\mathrm{q}^R(M, N) < \infty$ and CI-$\dim_R(M) < \infty$, then the derived depth formula holds for M and N; see [20, 5.3]. This recovers a result of Foxby [23, 2.1] who initially proved that the derived depth formula holds for M and N if $\mathrm{pd}_R(M) < \infty$; see also [28, 2.2]. $\quad\square$

We finish this section by recording the preliminary results which are needed for the proof of Theorem 3.4. To not to disturb the flow of the paper, we prove these preliminary results, namely 2.14, 2.15, and 2.16 in section 5.

2.14. Let M and N be R-modules. Assume there is a short exact sequence of R-modules
$$0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0,$$
where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers.

  (i) If $\mathrm{depth}_R(M) \leq \mathrm{depth}(R)$, then $\mathrm{depth}_R(M) = \mathrm{depth}_R(K)$.
  (ii) If $q^R(M,N) < \infty$, then $q^R(M,N) = q^R(K,N)$.
 (iii) Assume $q^R(M,N) < \infty$. If $n \geq 1$, or $\mathrm{depth}_R\left(\mathrm{Tor}^R_{q^R(M,N)}(M,N)\right) \leq 1$, then
$$\mathrm{depth}_R\left(\mathrm{Tor}^R_{q^R(M,N)}(M,N)\right) = \mathrm{depth}_R\left(\mathrm{Tor}^R_{q^R(K,N)}(K,N)\right).$$

2.15. Let M and N be R-modules. Assume $q^R(M,N) < \infty$. Assume further there exists $x \in \mathfrak{m}$ such that x is a non zero-divisor on R, M, and N. Then the following conditions are equivalent:

  (i) $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N)$.
  (ii) $\mathrm{depth}_{R/xR}(M/xM) + \mathrm{depth}_{R/xR}(N/xN) = \mathrm{depth}(R/xR) + \mathrm{depth}_{R/xR}(M/xM \otimes_{R/xR}^{\mathbb{L}} N/xN)$.

2.16. Let M and N be R-modules such that $q^R(M,N) < \infty$. Assume the following conditions hold:

  (i) $\mathrm{depth}_R(M) \leq \mathrm{depth}(R)$.
  (ii) There is an exact sequence of R-modules $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers, and the derived depth formula holds for K and N.

Then $\mathrm{depth}_R(K \otimes_R^{\mathbb{L}} N) = \min\{\mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N), \mathrm{depth}_R(N)\}$. $\square$

3. MAIN RESULT

In this section we prove Theorem 1.2 and determine some conditions that imply the derived depth formula holds. We start by preparing a lemma; recall that we set $q^R(M,N) = \sup\{i : \mathrm{Tor}_i^R(M,N) \neq 0\}$ for given R-modules M and N.

Lemma 3.1. *Let M and N be R-modules such that $q^R(M,N) < \infty$. Assume there is an exact sequence of R-modules $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers. Assume the derived depth formula holds for K and N. Assume further* at least *one of the following holds:*

  (i) $\mathrm{depth}_R(M) < \mathrm{depth}(R)$.
  (ii) $\mathrm{depth}_R(M) = \mathrm{depth}(R)$ *and N is maximal Cohen-Macaulay.*
 (iii) $\mathrm{depth}_R(M) = \mathrm{depth}(R)$ *and* $\mathrm{depth}_R(\mathrm{Tor}^R_{q^R(M,N)}(M,N)) \leq 1$.

Then the derived depth formula holds for M and N.

Proof. As we assume $\mathrm{depth}_R(M) \leq \mathrm{depth}(R)$, it follows from 2.14(i) that $\mathrm{depth}_R(K) = \mathrm{depth}_R(M)$. Also, by our hypothesis, the derived depth formula holds for K and N. These facts yield:

$$(3.1.1) \qquad \mathrm{depth}_R(N) - \mathrm{depth}(K \otimes_R^{\mathbb{L}} N) = \mathrm{depth}(R) - \mathrm{depth}_R(M).$$

So, in view of (3.1.1), the derived depth formula holds for M and N if $\mathrm{depth}(M \otimes_R^{\mathbb{L}} N) = \mathrm{depth}_R(K \otimes_R^{\mathbb{L}} N)$. Moreover, by 2.16, we have:

$$(3.1.2) \qquad \mathrm{depth}_R(K \otimes_R^{\mathbb{L}} N) = \min\{\mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N), \mathrm{depth}_R(N)\}.$$

Assume part (i) holds. Then (3.1.1) implies that $\mathrm{depth}_R(N) > \mathrm{depth}(K \otimes_R^{\mathbb{L}} N)$. Hence we conclude from (3.1.2) that $\mathrm{depth}_R(K \otimes_R^{\mathbb{L}} N) = \mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N)$.

Assume part (ii) holds. Then we have $\mathrm{depth}_R(N) = \dim(R)$. Moreover, as $q^R(M,N) < \infty$, it follows that $\mathrm{depth}(M \otimes_R^{\mathbb{L}} N) \leq \dim(R)$; see 2.13(ii). Thus (3.1.2) yields $\mathrm{depth}(K \otimes_R^{\mathbb{L}} N) = \mathrm{depth}(M \otimes_R^{\mathbb{L}} N)$.

Finally assume part (iii) holds. Note that $q^R(M,N) = q^R(K,N)$; see 2.14(ii). Set $q = q^R(M,N)$. As we assume $\mathrm{depth}_R(\mathrm{Tor}_q^R(M,N)) \leq 1$, 2.13(iii) shows that

$$(3.1.3) \qquad \mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N) = \mathrm{depth}_R(\mathrm{Tor}_q^R(M,N)) - q.$$

On the other hand, we know by 2.14(iii) that $\mathrm{depth}_R(\mathrm{Tor}_q^R(K,N)) = \mathrm{depth}_R(\mathrm{Tor}_q^R(M,N))$. Hence, using 2.13(iii) once more, we see:

$$(3.1.4) \qquad \mathrm{depth}_R(K \otimes_R^{\mathbb{L}} N) = \mathrm{depth}_R(\mathrm{Tor}_q^R(K,N)) - q.$$

Consequently (3.1.3) and (3.1.4) imply that $\mathrm{depth}(K \otimes_R^{\mathbb{L}} N) = \mathrm{depth}(M \otimes_R^{\mathbb{L}} N)$. $\qquad\square$

Proposition 3.2. *Let M and N be R-modules such that $\mathrm{q}^R(M,N) < \infty$. Assume at least one of the following conditions holds:*

(i) $\mathrm{depth}_R(M) < \mathrm{depth}(R)$.
(ii) $\mathrm{depth}_R(M) = \mathrm{depth}(R)$ *and N is maximal Cohen-Macaulay.*
(iii) $\mathrm{depth}_R(M) = \mathrm{depth}(R)$ *and $\mathrm{depth}_R(\mathrm{Tor}_q(M,N)) \leq 1$.*

If red-CI-$\dim_R(M) < \infty$, *then the derived depth formula holds for M and N.*

Proof. We set red-CI-$\dim_R(M) = c$ and proceed by induction on c. If $c = 0$, then CI-$\dim_R(M) < \infty$ so that the derived depth formula holds for M and N; see 2.3 and 2.13(vi).

Next assume $c \geq 1$. Then there is an exact sequence of R-modules $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers, such that red-CI-$\dim_R(K) <$ red-CI-$\dim_R(M)$; see 2.6(i). Moreover, 2.14 implies that the conditions (i), (ii), and (iii) hold for K. Hence, by the induction hypothesis, we see that the derived depth formula holds for K and N. Consequently, the result follows from Lemma 3.1. $\qquad\square$

Our next result, Theorem 3.4, establishes Theorem 1.2 and generalizes the result of Bergh-Jorgensen, which is stated as Theorem 1.1 in the introduction. Prior to proving Theorem 3.4, we recall a few related points, particularly concerning the depth formula and the derived depth formula.

Recall that, given R-modules M and N, if $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R N)$, then we say that the depth formula holds for M and N. Similarly, we say that the derived depth formula holds if $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N)$. If M and N are Tor-independent R-modules, then $\mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N) = \mathrm{depth}_R(M \otimes_R N)$ so that these two depth formulas are the same. However, in general, neither of them implies the other one; see the next example and also Example 3.9.

Example 3.3. Let R be a local ring of positive depth d, $x \in \mathfrak{m}$ be a non zero-divisor on R, $M = R/xR$, and $N = k$. Then the derived depth formula holds for M and N since $\mathrm{pd}_R(M) < \infty$; see 2.13(vi). This implies that $\mathrm{depth}_R(M \otimes_R^{\mathbb{L}} N) = -1$. On the other hand, since $\mathrm{depth}_R(N) = \mathrm{depth}_R(M \otimes_R N) = 0$ and $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = d - 1 \neq d = d + \mathrm{depth}_R(M \otimes_R N)$, the depth formula for M and N fails.

Note that the hypothesis on the module M in the next theorem is strictly weaker than that of Theorem 1.1; see Examples 2.9 and 2.10.

Theorem 3.4. *If R is a Cohen-Macaulay local ring and M and N are R-modules such that $\mathrm{q}^R(M,N) < \infty$ and* red-CI-$\dim_R(M) < \infty$, *then the derived depth formula holds for M and N.*

Proof. Assume R is a Cohen-Macaulay, $\mathrm{q}^R(M,N) < \infty$, and red-CI-$\dim_R(M) < \infty$. We may also assume both M and N are nonzero. Then it follows that red-$\mathrm{pd}_R(M) < \infty$; see 2.7(i).

If $\mathrm{depth}_R(M) < \mathrm{depth}(R)$, Proposition 3.2(i) shows that the derived depth formula holds for M and N. Hence we may assume $\mathrm{depth}_R(M) = \mathrm{depth}(R)$, that is, M is maximal Cohen-Macaulay. We set $d = \dim(R)$ and proceed by induction on d.

If $d \leq 1$, then Proposition 3.2(iii) shows that the derived depth formula holds for M and N, and establishes the base case of the induction on d. Next we assume $d \geq 2$.

Case 1: Assume $\mathrm{depth}_R(N) \geq 1$. Then there is an element $x \in \mathfrak{m}$ which is non zero-divisor on M, N, and R. One can easily show that $\mathrm{q}^{R/xR}(M/xM, N/xN) < \infty$; see, for example, [35, Page 140, Lemma 2(iii)]. Furthermore, we have that red-$\mathrm{pd}_{R/xR}(M/xM) \leq$ red-$\mathrm{pd}_R(M) < \infty$; see 2.7(ii). As R/xR is a Cohen-Macaulay local ring with $\dim(R/xR) = d - 1 < d$, the induction hypothesis on d applied to the

pair $(M/xM, N/xN)$ over R/xR shows that the derived depth formula holds for M/xM and N/xN over R/xR. Thus the derived depth formula holds for M and N due to 2.15.

Case 2: Assume $\operatorname{depth}_R(N) = 0$. To deal with this case, we set red-$\operatorname{pd}_R(M) = c$ and proceed by induction on c. If $c = 0$, then $\operatorname{pd}_R(M) < \infty$ so that the derived depth formula holds for M and N; see 2.3 and 2.13(vi). So we assume $c \geq 1$. Then it follows from 2.6(i) that there is a short exact sequence of R-modules $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers, such that red-$\operatorname{pd}_R(K) <$ red-$\operatorname{pd}_R(M)$. Then, in view of 2.14(ii), the induction hypothesis implies that the derived depth formula holds for K and N. Also we know that $d = \operatorname{depth}_R(M) = \operatorname{depth}_R(K)$; see 2.14(i). Hence it suffices to show $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) = \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N)$.

We observe that $\operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) = 0$ since the derived depth formula holds for K and N, that is,

$$d = d + 0 = \operatorname{depth}_R(K) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) = d + \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N).$$

As $\operatorname{depth}_R(M) \leq \operatorname{depth}(R) = \dim(R)$, and since the derived depth formula holds for K and N over R, we have by 2.16 that $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) \geq \min\{\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(N)\} = \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) = 0$. Therefore we aim to prove $0 = \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) \geq \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N)$.

Consider a syzygy short exact sequence

$$(3.4.1) \qquad\qquad 0 \to \Omega_R N \to F \to N \to 0,$$

where F is a (finitely generated) free R-module. As $\operatorname{depth}_R(F) = \operatorname{depth}(R) = d \geq 2 > 0 = \operatorname{depth}_R(N)$, it follows by the depth lemma that $\operatorname{depth}_R(\Omega_R N) = 1$. So we can use Case 1 with the pair $(M, \Omega_R N)$, and conclude that $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} \Omega_R N) = \operatorname{depth}_R(M) + \operatorname{depth}_R(\Omega_R N) - d = 1$ (recall that $\operatorname{depth}_R(M) = d$).

We apply the functor $M \otimes_R^{\mathbb{L}} -$ to the exact triangle induced from (3.4.1), and obtain the exact triangle:

$$(3.4.2) \qquad\qquad M \otimes_R^{\mathbb{L}} \Omega_R N \to M \otimes_R^{\mathbb{L}} F \to M \otimes_R^{\mathbb{L}} N \to (M \otimes_R^{\mathbb{L}} \Omega N)[1].$$

Suppose $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) \geq \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} \Omega_R N) = 1$. Then the derived depth lemma 2.13(iv)(a) gives the following contradiction

$$1 = \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} \Omega_R N) \geq \min\big\{ \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} F), \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) + 1 \big\} \geq 2.$$

Consequently, $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) < \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} \Omega_R N) = 1$, that is, $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) \leq 0$. This establishes the claim when $\operatorname{depth}_R(N) = 0$, and completes the proof of the theorem by induction on d. $\qquad\square$

We do not know whether the conclusion of Theorem 3.4 holds over rings that are not necessarily Cohen-Macaulay. Hence the following question is natural:

Question 3.5. Let R be a local ring and let M and N be R-modules. If red-CI-$\dim_R(M) < \infty$ and $q^R(M, N) < \infty$, then must the derived depth formula hold for M and N?

We proceed to elaborate on Question 3.5. More precisely, we give examples which showcase the necessity of the finiteness of $q^R(M, N)$ and red-CI-$\dim_R(M)$ in the question; see Examples 3.7 and 3.9.

Remark 3.6. Let M and N be R-modules and set $X = M \otimes_R^{\mathbb{L}} N$. Then 2.13(iii) yields that

$$q^R(M, N) - e \leq -\operatorname{depth}_R(X) \leq q^R(M, N) \quad \text{and} \quad q^R(M, N) = \infty \text{ if and only if } \operatorname{depth}_R(X) = -\infty.$$

Therefore, if the derived depth formula holds, that is, if $\operatorname{depth}_R(M) + \operatorname{depth}_R(N) - \operatorname{depth}(R) = \operatorname{depth}_R(X)$, then $q^R(M, N) < \infty$. In other words, if $q^R(M, N) = \infty$, then the derived depth formula for M and N fails.

Example 3.7. Let $R = k[\![t^3, t^4, t^5]\!]$ and assume $|k| = \infty$, for example, $k = \mathbb{C}$. Then R is a non-regular Cohen-Macaulay local ring with minimal multiplicity. So, $q^R(k, k) = \infty$ and red-CI-$\dim_R(k) < \infty$; see [16, 2.14]. Moreover, by Remark 3.6, the derived depth formula fails for the pair (k, k).

Next we record an example of a local ring R and R-modules M and N such that $q^R(M, N) < \infty$, red-CI-$\dim_R(M) = \infty = $ red-CI-$\dim_R(N)$, and the derived depth formula for M and N fails. The example we seek can be obtained as an immediate consequence of a construction of Jorgensen-Şega [32]:

3.8. It is obtained in [32, 3.3] that there exists an Artinian Gorenstein local ring R and modules M and T over R such that $\mathrm{Ext}^1_R(M,T) \neq 0 = \mathrm{Ext}^i_R(M,T)$ for all $i \geq 2$ (in the notations of [32], we set $R = A_\alpha$ and $T = T_1$; see [32, 3.1 and pages 475, 476, and 479] for these definitions and the construction of M).

Example 3.9. Let R, M, and T be as in 3.8. If red-CI-$\dim_R(M) < \infty$, then $\mathrm{Ext}^1_R(M,T) = 0$ by Proposition 4.9. So red-CI-$\dim_R(M) = \infty$. Note that $\mathrm{Ext}^i_R(M,T) \cong \mathrm{Ext}^i_R(T^*,M^*)$ for all $i \geq 1$; see, for example, [26, 2.2]. Therefore, we can use Proposition 4.9 once more and see that red-CI-$\dim_R(T^*) = \infty$. Set $N = T^*$. Then $\mathrm{q}^R(M,N) = 1 < \infty$; see, for example [31, 2.7].

Next suppose $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes^{\mathbb{L}}_R N)$, that is, the derived depth formula for M and N holds. As R is Artinian, it follows that $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) - \mathrm{depth}(R) = 0$. Thus $\mathrm{depth}_R(M \otimes^{\mathbb{L}}_R N) = 0$. On the other hand, setting $X = M \otimes^{\mathbb{L}}_R N$, we have that $s = \sup(X) = \mathrm{q}^R(M,N) = 1$ and $\mathrm{depth}_R(\mathrm{H}_s(X)) = 0$. Hence, 2.13(iii) shows that $\mathrm{depth}_R(X) = \mathrm{depth}_R(\mathrm{H}_s(X)) - s = -1$. Consequently, the derived depth formula for M and N fails (let us also note that the depth formula for M and N holds trivially since R is Artinian.)

Remark 3.10. As it relates to Theorem 1.2 and to Question 3.5, we also mention the following result established in [34]: There exists a Gorenstein local ring R and Tor-independent R-modules M and N such that the depth formula (or equivalently, the derived depth formula in this context) does not hold for M and N. This example relies on a construction by Jorgensen-Şega [33]; see [34] for the details.

4. SOME COROLLARIES AND FURTHER RESULTS

In this section we obtain some corollaries of Proposition 3.2 and Theorem 3.4, and also consider further consequences of our results from sections 2 and 3.

Auslander [4] proved the following general version of the depth formula: If R is a local ring, and M and N are R-modules such that $\mathrm{q}^R(M,N) < \infty$, then

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) - \mathrm{q}^R(M,N)$$

provided that $\mathrm{q}^R(M,N) = 0$, or $\mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) \leq 1$. This version of the depth formula was also studied in the literature. One such result is due to Bergh-Jorgensen; see [12, page 3] for the definition of upper reducing degree reddeg$^*M$ of M.

4.1. (Bergh-Jorgensen [12, 3.1]; see also [11, 3.4(i)]) Let R be a Cohen-Macaulay local ring and let M and N be R-modules such that $\mathrm{q}^R(M,N) < \infty$. Assume at least one of the following conditions holds:

(i) $\mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) = 0$.

(ii) $\mathrm{q}^R(M,N) \geq 1$, $\mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) \leq 1$, and reddeg$^*M \geq 2$.

If M has reducible complexity, then

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) - \mathrm{q}^R(M,N).$$

Next, as a consequence of Proposition 3.2(iii) and Theorem 3.4, we generalize 4.1:

Corollary 4.2. *Let R be a Cohen-Macaulay local ring and let M and N be R-modules such that $\mathrm{q}^R(M,N) < \infty$ and $\mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) \leq 1$.*

If red-CI-$\dim_R(M) < \infty$ *(for example, M has reducible complexity), then*

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right) - \mathrm{q}^R(M,N).$$

Proof. We set $X = M \otimes^{\mathbb{L}}_R N$ and $q = \mathrm{q}^R(M,N)$. Then $\mathrm{depth}_R(\mathrm{H}_q(X)) = \mathrm{depth}_R\left(\mathrm{Tor}^R_q(M,N)\right) \leq 1$ and $\sup(X) = q < \infty$. Hence 2.13(iii) implies that $\mathrm{depth}_R(X) = \mathrm{depth}_R\left(\mathrm{Tor}^R_q(M,N)\right) - q$. As we know by Proposition 3.2(iii) that the derived depth formula holds for M and N, the claim follows. □

Remark 4.3. Sadeghi [36, 3.2(ii)] proved that the conclusion of Corollary 4.2 holds over local rings that are not necessarily Cohen-Macaulay provided that $1 \leq q^R(M,N) < \infty$ and M has reducible complexity. Corollary 4.2 is also due to Bergh [11, 3.4(i)] in case $\mathrm{depth}_R\left(\mathrm{Tor}^R_{q^R(M,N)}(M,N)\right) = 0$ and M has reducible complexity. ◻

We proceed and obtain some consequences of our results from sections 2 and 3. These consequences do not directly imply the depth formula holds, but they contribute to the study of depth of tensor products of modules.

For their study of the depth formula, Bergh-Jorgensen [12] proved the following results.

4.4. (Bergh-Jorgensen [12, 3.2 and 3.7]) Let R be a Cohen-Macaulay local ring and let M and N be nonzero R-modules such that M and N are Tor-independent, that is, $q^R(M,N) = 0$. Assume M has reducible complexity. Assume further $\mathrm{depth}_R(M \otimes_R N) \neq 0$. Then the following hold:

(i) $\mathrm{depth}_R(M) \neq 0$.
(ii) If M is maximal Cohen-Macaulay and G-$\mathrm{dim}_R(N) < \infty$, then $\mathrm{depth}_R(N) \neq 0$.   ◻

Thanks to Theorem 3.4, we can generalize 4.4 as follows:

Corollary 4.5. *Let R be a Cohen-Macaulay local ring and let M and N be nonzero R-modules such that M and N are Tor-independent, that is, $q^R(M,N) = 0$. Assume red-CI-$\mathrm{dim}_R(M) < \infty$ (for example, M has reducible complexity). Assume further $\mathrm{depth}_R(M \otimes_R N) \neq 0$. Then $\mathrm{depth}_R(M) \neq 0$ and $\mathrm{depth}_R(N) \neq 0$.*

Proof. We know by Theorem 3.4 that $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes^{\mathbb{L}}_R N)$. As $q^R(M,N) = 0$, it follows that $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R N)$. Therefore, if $\mathrm{depth}_R(M) = 0$, then $\mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R N) > \mathrm{depth}(R) = \mathrm{dim}(R)$, which is not possible. Hence $\mathrm{depth}_R(M) \neq 0$. Similarly, it follows that $\mathrm{depth}_R(N) \neq 0$.   ◻

The preliminary results recorded in section 2 yield new results on the vanishing of Ext and Tor. As a demonstration, we use 2.7 and 2.14, and prove the following; see [11, 3.3] for a similar result for modules with reducible complexity.

Proposition 4.6. *Let M and N be nonzero R-modules such that $q^R(M,N) < \infty$.*

(i) *If* red-CI-$\mathrm{dim}_R(M) < \infty$, *then* $q^R(M,N) \leq \mathrm{depth}(R)$.
(ii) *If* red-CI-$\mathrm{dim}_R(M) < \infty$ *and* $\mathrm{depth}_R(M) \leq \mathrm{depth}(R)$, *then*

$$\mathrm{depth}(R) - \mathrm{depth}_R(M) - \mathrm{depth}_R(N) \leq q^R(M,N) \leq \mathrm{depth}(R) - \mathrm{depth}_R(M).$$

Proof. Assume red-CI-$\mathrm{dim}_R(M) < \infty$. It follows from 2.7(i) that red-$\mathrm{pd}_R(M) < \infty$. Let $\{K_0, \ldots, K_r\}$ be a reducing pd-sequence of M; see 2.3. Then $\mathrm{pd}(K_r) < \infty$ and $q^R(M,N) = q^R(K_r,N)$; see 2.14(ii). Therefore, part (i) follows since we have

(4.6.1) $q^R(M,N) = q^R(K_r,N) \leq \mathrm{pd}_R(K_r) = \mathrm{depth}(R) - \mathrm{depth}_R(K_r) \leq \mathrm{depth}(R).$

Next assume $\mathrm{depth}_R(M) \leq \mathrm{depth}(R)$. Then $\mathrm{depth}_R(M) = \mathrm{depth}(K_r)$ so that (4.6.1) yields the inequality $q^R(M,N) \leq \mathrm{depth}(R) - \mathrm{depth}_R(M)$; see 2.14(i). As $\mathrm{pd}_R(K_r) < \infty$, the derived depth formula holds for the pair (K_r, N); see 2.13(vi). Hence we have:

(4.6.2) $\mathrm{depth}_R(K_r \otimes^{\mathbb{L}}_R N) = \mathrm{depth}_R(K_r) + \mathrm{depth}_R(N) - \mathrm{depth}(R).$

Furthermore 2.13(iii) implies that:

(4.6.3) $\mathrm{depth}_R(K_r \otimes^{\mathbb{L}}_R N) \geq -q^R(M,N).$

Thus (4.6.2) and (4.6.3) show that $q^R(M,N) \geq \mathrm{depth}(R) - \mathrm{depth}_R(M) - \mathrm{depth}_R(N)$. Consequently, part (ii) follows.   ◻

We record some corollories of Proposition 4.6. Part (i) of the next corollary is known by [30, 2.3] if CI-$\mathrm{dim}_R(M) < \infty$, while part (ii) follows from [6, 1.1] if $\mathrm{pd}_R(M) < \infty$.

Corollary 4.7. *Let M and N be R-modules such that $\mathrm{q}^R(M,N) < \infty$ and red-CI-dim$_R(M) < \infty$.*

  (i) *If $\mathrm{depth}(R) = \mathrm{depth}_R(M)$, then $\mathrm{q}^R(M,N) = 0$.*
 (ii) *If $\mathrm{depth}(R) \geq \mathrm{depth}_R(M)$ and $\mathrm{depth}_R(N) = 0$, then $\mathrm{q}^R(M,N) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$.*

Proof. The claims follow immediately from Proposition 4.6(ii). $\qquad\square$

Remark 4.8. Let M and N be R-modules such that $\mathrm{q}^R(M,N) < \infty$ and CI-dim$_R(M) < \infty$. It follows that $\mathrm{q}^R(M,N) \leq$ CI-dim$_R(M) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$; see [30, 2.3]. Moreover, if $\mathrm{depth}_R(N) = 0$, then $\mathrm{q}^R(M,N) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$ by, for example, Corollary 4.7.

Let M and N be R-modules such that $N \neq 0$ and $\mathrm{p}^R(M,N) = \sup\{i : \mathrm{Ext}^i_R(M,N) \neq 0\}$. Assume red-pd$_R(M) < \infty$ and $\{K_0, \ldots, K_r\}$ is a reducing pd-sequence of M. Then it can be proven along the same lines as in 2.14(ii) that $\mathrm{p}^R(M,N) = \mathrm{p}^R(K_r,N)$. As $\mathrm{pd}_R(K_r) < \infty$, it follows from [8, 4.10] that $\mathrm{p}^R(K_r,N) = \mathrm{depth}(R) - \mathrm{depth}_R(K_r)$. So, by 2.14(i), the following result holds:

Proposition 4.9. *Let M and N be R-modules such that $N \neq 0$, $\mathrm{depth}_R(M) \leq \mathrm{depth}(R)$ and $\mathrm{p}^R(M,N) < \infty$. If red-CI-dim$_R(M) < \infty$, then $\mathrm{p}^R(M,N) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$.*

Remark 4.10. Proposition 4.9 allows us to find examples of modules that do not have finite reducing complete intersection dimension. For example, Jorgensen and Sega [32, page 475, 476 and 3.3] proved that there exists an Artinian Gorenstein local ring R and R-modules M and N such that $\mathrm{Ext}^1_R(M,N) \neq 0$ and $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \geq 2$. Therefore Proposition 4.9 implies that red-CI-dim$_R(M) = \infty$. $\qquad\square$

Recall that an R-module M is torsion, that is, M equals its torsion submodule, if and only if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$. The next result is to be compared with [17, A2].

Corollary 4.11. *Let M and N be R-modules. Assume red-CI-dim$_R(M) < \infty$. If $\mathrm{Tor}^R_i(M,N)$ is torsion for all $i \gg 0$, then $\mathrm{Tor}^R_i(M,N)$ is torsion for all $i \geq 1$.*

Proof. Assume $\mathrm{Tor}^R_i(M,N)$ is torsion for all $i \gg 0$, and let $\mathfrak{p} \in \mathrm{Ass}(R)$. Then $\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) < \infty$. Also red-CI-dim$_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ by Lemma 2.5. So Proposition 4.6(i) shows that $\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \mathrm{depth}(R_{\mathfrak{p}}) = 0$, that is, $\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$. This proves that $\mathrm{Tor}^R_i(M,N)$ is torsion for all $i \geq 1$. $\qquad\square$

Corollary 4.12. *Let R be a one-dimensional Cohen-Macaulay local ring and let M and N be nonzero R-modules. Assume $\mathrm{q}^R(M,N) < \infty$ and red-CI-dim$_R(M) < \infty$. Then the following are equivalent:*

  (i) *$\mathrm{q}^R(M,N) = 0$.*
 (ii) *M or N is torsion-free.*
(iii) *The depth formula holds, that is, $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R N)$.*

Proof. It follows from Theorem 3.4 that part (i) implies part (iii). Now assume part (iii) holds. Then we have that $2 \geq \mathrm{depth}_R(M) + \mathrm{depth}_R(N) = 1 + \mathrm{depth}_R(M \otimes_R N) \geq 1$. So $\mathrm{depth}_R(M) = 1$ or $\mathrm{depth}_R(N) = 1$, that is, M or N is torsion-free. Hence part (iii) implies part (ii).

Next assume part (ii) holds. We know $\mathrm{q}^R(M,N) \leq 1$; see Proposition 4.6(i). Suppose $\mathrm{q}^R(M,N) \neq 0$, that is, $\mathrm{q}^R(M,N) = 1$. As $\mathrm{depth}_R(\mathrm{Tor}^R_1(M,N)) \leq 1$, Corollary 4.2 yields

$$(4.12.1) \qquad \mathrm{depth}_R(M) + \mathrm{depth}_R(N) = 1 + \mathrm{depth}_R(\mathrm{Tor}^R_1(M,N)) - 1 = \mathrm{depth}_R(\mathrm{Tor}^R_1(M,N)).$$

It follows from Corollary 4.11 that $\mathrm{Tor}^R_1(M,N)$ is torsion, that is, $\mathrm{Tor}^R_1(M,N)$ has finite length. Therefore (4.12.1) gives the equality $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = 0$. This is not possible since M or N is torsion-free. Consequently $\mathrm{q}^R(M,N) = 0$ so that part (ii) implies part (i). $\qquad\square$

Remark 4.13. In Corollary 4.12, M and N are not necessarily both torsion-free. If R is a one-dimensional Cohen-Macaulay local ring, $M = R/xR$ for some non zero-divisor x on R, and N is a torsion-free R-module, then M is torsion, $\mathrm{q}^R(M,N) = 0$, and $\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(M \otimes_R N)$.

Next we consider the dependency formula of Jorgensen which is a generalization of the classical Auslander-Buchsbaum formula (recall that $\mathrm{depth}_R(0) = \infty$).

4.14. (Jorgensen [29, 2.7]) Let M and N be nonzero R-modules. If $\mathrm{q}^R(M,N) < \infty$ and $\text{CI-dim}_R(M) < \infty$, then it follows:

$$\mathrm{q}^R(M,N) = \sup\left\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Supp}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right)\right\}.$$

Our next result, namely Theorem 4.15, generalizes 4.14 over Cohen-Macaulay rings and establishes Theorem 1.3. Note that Theorem 4.15 not only improves 4.14 by replacing the assumption $\text{CI-dim}_R(M) < \infty$ with $\text{red-CI-dim}_R(M) < \infty$, but also it points out that $\mathrm{q}^R(M,N)$ can be computed by using the (finitely many) prime ideals in $\mathrm{Ass}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right)$.

Theorem 4.15. *Let R be a Cohen-Macaulay local ring and let M and N be nonzero R-modules. If $\mathrm{q}^R(M,N) < \infty$ and $\text{red-CI-dim}_R(M) < \infty$, then it follows:*

$$\mathrm{q}^R(M,N) = \sup\left\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\right\}$$
$$= \sup\left\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Ass}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right)\right\}.$$

Proof. We know $\text{red-pd}_R(M) < \infty$ since we assume $\text{red-CI-dim}_R(M) < \infty$; see 2.7(i). Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then $\text{red-pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{red-pd}_R(M) < \infty$ by Lemma 2.5. Set $X = M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$. Then it follows that $\sup(X) = \mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \mathrm{q}^R(M,N) < \infty$. Thus, by 2.13(iii), we have that $\mathrm{depth}_R(X) \geq -\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$. This yields:

$$(4.15.1) \qquad \mathrm{q}^R(M,N) \geq \mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \geq -\mathrm{depth}(M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}).$$

As $R_{\mathfrak{p}}$ is Cohen-Macaulay and $\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) < \infty$, Theorem 3.4 implies that the derived depth formula holds for $(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ over $R_{\mathfrak{p}}$, that is, $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) + \mathrm{depth}(M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})$. Hence (4.15.1) implies that:

$$(4.15.2) \qquad \mathrm{q}^R(M,N) \geq \mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}).$$

So $\mathrm{q}^R(M,N) \geq \mathrm{depth}(R_{\mathfrak{q}}) - \mathrm{depth}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) - \mathrm{depth}_{R_{\mathfrak{q}}}(N_{\mathfrak{q}})$ for each $\mathfrak{q} \in \mathrm{Spec}(R)$; see (4.15.2). Consequently we conclude that

$$(4.15.3) \quad \mathrm{q}^R(M,N) \geq \sup\left\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\right\}$$
$$\geq \sup\left\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Ass}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right)\right\}.$$

Next let $\mathfrak{p} \in \mathrm{Ass}\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right)$. Then $\mathrm{depth}_{R_{\mathfrak{p}}}\left(\mathrm{Tor}^{R_{\mathfrak{p}}}_{\mathrm{q}^R(M,N)}(M_{\mathfrak{p}}, N_{\mathfrak{p}})\right) = 0$. As $\mathrm{depth}_R(0) = \infty$, it follows that $\mathrm{Tor}^{R_{\mathfrak{p}}}_{\mathrm{q}^R(M,N)}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0$. Thus we deduce $\mathrm{q}^R(M,N) = \mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$. Set $Y = M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$. Then $\sup(Y) = \mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) < \infty$ and $\mathrm{depth}_{R_{\mathfrak{p}}}\left(\mathrm{H}_{\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})}(Y)\right) = \mathrm{depth}_{R_{\mathfrak{p}}}\left(\mathrm{Tor}^{R_{\mathfrak{p}}}_{\mathrm{q}^R(M,N)}(M_{\mathfrak{p}}, N_{\mathfrak{p}})\right) = 0$. So 2.13(iii) yields:

$$(4.15.4) \qquad \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}) = \mathrm{depth}_{R_{\mathfrak{p}}}(Y) = \mathrm{depth}_{R_{\mathfrak{p}}}\left(\mathrm{H}_{\mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})}(Y)\right) - \mathrm{q}^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}).$$

Recall, by Theorem 3.4, we have $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) + \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})$. So we conclude from (4.15.4) that

$$(4.15.5) \qquad \mathrm{q}^R(M,N) = -\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes^{\mathbb{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}).$$

As the equality in (4.15.5) is true for each associated prime ideal of $\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)$, we deduce that

$$(4.15.6) \quad \mathrm{q}^R(M,N) = \sup\left\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Ass}_R\left(\mathrm{Tor}^R_{\mathrm{q}^R(M,N)}(M,N)\right)\right\}.$$

Now (4.15.3) and (4.15.6) give the required conclusion.   □

5. PROOFS OF 2.14, 2.15, AND 2.16

This section is devoted to the proofs of the preliminary results stated in section 2. Recall that R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and all R-modules are assumed to be finitely generated. Moreover, we define an R-complex as a complex of finitely generated R-modules which is indexed homologically.

Proof of 2.14. Part (i) follows by the depth lemma so we proceed with the proofs of parts (ii) and (iii). Set $q = q^R(M,N)$. The short exact sequence $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$ induces a long exact sequence in Tor modules, which implies the vanishing of $\operatorname{Tor}_i^R(K,N)$ for all $i \geq q+1$ and also yields the exact sequence:

$$(2.14.1) \qquad 0 \to \operatorname{Tor}_q^R(M^{\oplus a},N) \to \operatorname{Tor}_q^R(K,N) \to \operatorname{Tor}_q^R(\Omega_R^n M^{\oplus b},N).$$

Hence, as $\operatorname{Tor}_q^R(M,N) \neq 0$, it follows that $\operatorname{Tor}_q^R(K,N) \neq 0$. This proves $q = q^R(K,N)$.

Assume $n \geq 1$. Then $\operatorname{Tor}_q^R(\Omega_R^n M^{\oplus b},N) = 0$. Hence (2.14.1) yields $\operatorname{Tor}_q^R(M^{\oplus a},N) \cong \operatorname{Tor}_q^R(K,N)$ and shows that $\operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right) = \operatorname{depth}_R\left(\operatorname{Tor}_q^R(K,N)\right)$, as required.

Next we assume $\operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right) \leq 1$ and $n = 0$. We observe that (2.14.1) yields two exact sequences:

$$(2.14.2) \qquad 0 \to \operatorname{Tor}_q^R(M^{\oplus a},N) \to \operatorname{Tor}_q^R(K,N) \to X \to 0 \ \text{ and } \ 0 \to X \to \operatorname{Tor}_q^R(M^{\oplus a},N).$$

If $\operatorname{depth}_R(\operatorname{Tor}_q^R(M,N)) = 0$, then the leftmost exact sequence in (2.14.2) implies that the depth of $\operatorname{Tor}_q^R(K,N)$ is zero. Thus we may assume $\operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right) = 1$. In that case $\operatorname{depth}_R(X) \geq 1$ due to the rightmost exact sequence in (2.14.2). So it follows that

$$\operatorname{depth}_R\left(\operatorname{Tor}_q^R(K,N)\right) \geq \min\{\operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right), \operatorname{depth}_R(X)\} = 1.$$

If $\operatorname{depth}_R\left(\operatorname{Tor}_q^R(K,N)\right) \geq 2 = 1 + \operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right)$, then the depth lemma applied to the leftmost exact sequence in (2.14.2) implies that $\operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right) = \operatorname{depth}_R(X) + 1 \geq 2$, which is not true. Consequently, we conclude that $\operatorname{depth}_R\left(\operatorname{Tor}_q^R(K,N)\right) = 1 = \operatorname{depth}_R\left(\operatorname{Tor}_q^R(M,N)\right)$.   □

We make use of the following lemma for the proof of 2.15.

Lemma 5.1. *Let Y be a homologically bounded R-complex, $x \in \mathfrak{m}$, and let $K = (0 \to R \xrightarrow{x} R \to 0)$ be the Koszul complex on x. Then $\operatorname{depth}_R(Y \otimes_R^{\mathbb{L}} K) = \operatorname{depth}_R(Y) - 1$.*

Proof. Note that we have the exact triangle $Y \xrightarrow{x} Y \to Y \otimes_R^{\mathbb{L}} K \to Y[1]$. Given $i \geq 0$, this exact triangle yields the following exact sequence of R-modules:

$$(5.1.1) \qquad \operatorname{Ext}_R^{i-1}(k,Y) \to \operatorname{Ext}_R^{i-1}(k,Y \otimes_R^{\mathbb{L}} K) \to \operatorname{Ext}_R^i(k,Y) \xrightarrow{x} \operatorname{Ext}_R^i(k,Y).$$

Set $n = \operatorname{depth}_R(Y)$. Then, as the map on $\operatorname{Ext}_R^n(k,Y)$ given by multiplication by x is zero, (5.1.1) implies that $\operatorname{Ext}_R^{n-1}(k,Y \otimes_R^{\mathbb{L}} K) \cong \operatorname{Ext}_R^n(k,Y)$. So $\operatorname{Ext}_R^{n-1}(k,Y \otimes_R^{\mathbb{L}} K) \neq 0$. Also, if $i \leq n-2$, then (5.1.1) shows that $\operatorname{Ext}_R^i(k,Y \otimes_R^{\mathbb{L}} K) = 0$. Thus $\operatorname{depth}_R(Y \otimes_R^{\mathbb{L}} K) = -\sup \operatorname{RHom}_R(k,Y \otimes_R^{\mathbb{L}} K) = n-1$.   □

Proof of 2.15. We may assume M and N are nonzero. Hence, since $q^R(M,N) < \infty$, depths of the complexes in parts (i) and (ii) are finite; see 2.13(ii) and 2.14(iii). Set $S = R/xR$. Then we have the following quasi-isomorphisms:

$$(M \otimes_R^{\mathbb{L}} N) \otimes_R^{\mathbb{L}} S \simeq (M \otimes_R^{\mathbb{L}} S) \otimes_R^{\mathbb{L}} N \simeq \left((M \otimes_R^{\mathbb{L}} S) \otimes_S^{\mathbb{L}} S\right) \otimes_R^{\mathbb{L}} N$$

$$(2.15.1) \qquad\qquad\qquad\qquad \simeq (M \otimes_R^{\mathbb{L}} S) \otimes_S^{\mathbb{L}} (N \otimes_R^{\mathbb{L}} S)$$

$$\simeq (M/xM) \otimes_S^{\mathbb{L}} (N/xN)$$

Here the last quasi-isomorphism in (2.15.1) holds since $M \otimes_R^{\mathbb{L}} S \simeq M/xM$ and $N \otimes_R^{\mathbb{L}} S \simeq N/xN$ due to the fact that x is a non-zero divisor on R, M, and N.

Now (2.15.1) and Lemma 5.1 yield

$$(2.15.2) \qquad \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) - 1 = \operatorname{depth}_R\left((M \otimes_R^{\mathbb{L}} N) \otimes_R^{\mathbb{L}} S\right) = \operatorname{depth}_R\left((M/xM) \otimes_S^{\mathbb{L}} (N/xN)\right)$$
$$= \operatorname{depth}_S\left((M/xM) \otimes_S^{\mathbb{L}} (N/xN)\right).$$

Here the last equality in (2.15.2) holds as $\operatorname{depth}_S(X) = \operatorname{depth}_R(X)$ for each S-complex X; see [28, 5.2(1)]. Moreover, we know $\operatorname{depth}_S(M/xM) = \operatorname{depth}_R(M) - 1$ and $\operatorname{depth}_S(N/xN) = \operatorname{depth}_R(N) - 1$; see, for example, [13, 1.2.10(d)]. Consequently we use (2.15.2) and deduce

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) \iff$$
$$\operatorname{depth}_R(M) - 1 + \operatorname{depth}_R(N) - 1 = \operatorname{depth}(R) - 1 + \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) - 1 \iff$$
$$\operatorname{depth}_S(M/xM) + \operatorname{depth}_S(N/xN) = \operatorname{depth}(S) + \operatorname{depth}_S(M/xM \otimes_S^{\mathbb{L}} N/xN).$$

This proves the equivalence of part (i) and part (ii). □

To prove 2.16, we first make an observation:

Remark 5.2. Let M and N be R-modules such that $q^R(M,N) < \infty$.

Consider the syzygy short exact sequence $0 \to \Omega_R M \to F \to M \to 0$, where F is a free R-module. This sequence yields the exact triangle

$$(5.2.1) \qquad\qquad \Omega_R M \otimes_R^{\mathbb{L}} N \to F \otimes_R^{\mathbb{L}} N \to M \otimes_R^{\mathbb{L}} N \to (\Omega_R M \otimes_R^{\mathbb{L}} N)[1].$$

As $\operatorname{depth}_R(F \otimes_R^{\mathbb{L}} N) = \operatorname{depth}_R(N)$ and $\operatorname{depth}_R\left((M \otimes_R^{\mathbb{L}} N)[-1]\right) = \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) + 1$, the derived depth lemma 2.13(iv)(a) applied to (5.2.1) yields

$$\operatorname{depth}_R(\Omega_R M \otimes_R^{\mathbb{L}} N) \geq \min\left\{\operatorname{depth}_R(N), \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) + 1\right\}.$$

This argument shows that, for each integer $n \geq 1$, we can use induction on n and deduce:

$$\operatorname{depth}_R(\Omega_R^n M \otimes_R^{\mathbb{L}} N) \geq \min\left\{\operatorname{depth}_R(N), \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) + n\right\}.$$

Now we can give a proof of 2.16.

Proof of 2.16. The given exact sequence $0 \to M^{\oplus a} \to K \to \Omega_R^n M^{\oplus b} \to 0$ yields the exact triangle:

$$(2.16.1) \qquad\qquad (M \otimes_R^{\mathbb{L}} N)^{\oplus a} \to K \otimes_R^{\mathbb{L}} N \to (\Omega_R^n M \otimes_R^{\mathbb{L}} N)^{\oplus b} \to (M \otimes_R^{\mathbb{L}} N)^{\oplus a}[1].$$

Note, as $q^R(M,N) < \infty$, depths of the complexes in (2.16.1) are all finite; see 2.13(ii) and 2.14(i). The derived depth lemma 2.13(iv)(b) applied to (2.16.1) establishes the first inequality of the following:

$$\operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) \geq \min\{\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(\Omega_R^n M \otimes_R^{\mathbb{L}} N)\}$$
$$(2.16.2) \qquad\qquad\qquad \geq \min\{\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(N), \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) + n\}$$
$$= \min\{\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(N)\}.$$

Here, in (2.16.2), the second inequality holds due to Remark 5.2.

As $\operatorname{depth}_R(M) \leq \operatorname{depth}(R)$, we have $\operatorname{depth}_R(K) = \operatorname{depth}_R(M)$ so that $\operatorname{depth}_R(K) \leq \operatorname{depth}(R)$; see 2.14(i). Then, since the derived depth formula holds for K and N, we obtain

$$(2.16.3) \qquad \operatorname{depth}_R(N) = \operatorname{depth}(R) - \operatorname{depth}_R(K) + \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) \geq \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N).$$

The derived depth lemma 2.13(iv)(a) applied to (2.16.1) establishes the first inequality of:

$$\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) \geq \min\{\operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(\Omega_R^n M \otimes_R^{\mathbb{L}} N) + 1\}$$

$$(2.16.4) \qquad \geq \min\{\operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N) + n + 1, \operatorname{depth}_R(N) + 1\}$$

$$= \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N).$$

Here, in (2.16.4), we obtain the second inequality and the equality by using Remark 5.2 and (2.16.3), respectively. Now, since $\operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N)$ cannot exceed both $\operatorname{depth}_R(N)$ and $\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N)$, we conclude that $\min\{\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(N)\} \geq \operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N)$; see (2.16.3) and (2.16.4). Therefore, in view of (2.16.2), we see that $\operatorname{depth}_R(K \otimes_R^{\mathbb{L}} N) = \min\{\operatorname{depth}_R(M \otimes_R^{\mathbb{L}} N), \operatorname{depth}_R(N)\}$. $\qquad\square$

ACKNOWLEDGEMENTS

REFERENCES

[1] Tokuji Araya and Olgur Celikbas. Reducing invariants and total reflexivity. *Illinois J. Math.*, 64(2):169–184, 2020. 2, 3, 4, 5

[2] Tokuji Araya and Ryo Takahashi. On reducing homological dimensions over Noetherian rings. *Proc. Amer. Math. Soc.*, 150(2):469–480, 2022. 3, 4

[3] Tokuji Araya and Yuji Yoshino. Remarks on a depth formula, a grade inequality and a conjecture of Auslander. *Comm. Algebra*, 26(11):3793–3806, 1998. 1

[4] Maurice Auslander. Modules over unramified regular local rings. *Illinois J. Math.*, 5:631–647, 1961. 1, 9

[5] Maurice Auslander. Modules over unramified regular local rings. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 230–233. Inst. Mittag-Leffler, Djursholm, 1963. 3

[6] Maurice Auslander. *Anneaux de Gorenstein, et torsion en algèbre commutative*, volume 1966/67 of *Séminaire d'Algèbre Commutative dirigé par Pierre Samuel*. Secrétariat mathématique, Paris, 1967. 10

[7] Maurice Auslander and Mark Bridger. *Stable module theory*. Memoirs of the American Mathematical Society, No. 94. American Mathematical Society, Providence, R.I., 1969. 2

[8] Maurice Auslander and Oscar Goldman. Maximal orders. *Trans. Amer. Math. Soc.*, 97:1–24, 1960. 11

[9] Luchezar L. Avramov. Modules of finite virtual projective dimension. *Invent. Math.*, 96(1):71–101, 1989. 3

[10] Luchezar L. Avramov, Vesselin N. Gasharov, and Irena V. Peeva. Complete intersection dimension. *Inst. Hautes Études Sci. Publ. Math.*, (86):67–114 (1998), 1997. 2, 3

[11] Petter Andreas Bergh. Modules with reducible complexity. *J. Algebra*, 310:132–147, 2007. 1, 2, 3, 9, 10

[12] Petter Andreas Bergh and David A. Jorgensen. The depth formula for modules with reducible complexity. *Illinois J. Math.*, 55(2):465–478, 2011. 1, 9, 10

[13] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993. 5, 14

[14] Olgur Celikbas and Hailong Dao. Necessary conditions for the depth formula over Cohen-Macaulay local rings. *J. Pure Appl. Algebra*, 218(3):522 – 530, 2014. 1

[15] Olgur Celikbas, Souvik Dey, Toshinori Kobayashi, and Hiroki Matsui. Some characterizations of local rings via reducing dimensions. *To appear in Kyoto J. Math. Preprint 2022; posted at arXiv:2212.05220.* 4

[16] Olgur Celikbas, Souvik Dey, Toshinori Kobayashi, and Hiroki Matsui. On the reducing projective dimension over local rings. *Glasg. Math. J.*, 66(1):104–118, 2024. 2, 3, 4, 5, 8

[17] Olgur Celikbas, Srikanth B. Iyengar, Greg Piepmeyer, and Roger Wiegand. Criteria for vanishing of Tor over complete intersections. *Pacific J. Math.*, 276(1):93–115, 2015. 11

[18] Olgur Celikbas, Arash Sadeghi, and Naoki Taniguchi. On modules with reducible complexity. *Algebr. Represent. Theory*, 23(4):1467–1476, 2020. 1

[19] Sangki Choi and Srikanth Iyengar. On a depth formula for modules over local rings. *Comm. Algebra*, 29(7):3135–3143, 2001. 1

[20] Lars Winther Christensen and David A. Jorgensen. Vanishing of Tate homology and depth formulas over local rings. *J. Pure Appl. Algebra*, 219(3):464–481, 2015. 1, 5

[21] Nilkantha Das and Sutapa Dey. Some remarks on two-periodic modules over local rings. *Preprint 2023; posted at arXiv:2307.12752.* 1

[22] Hans-Bjørn Foxby. Bounded complexes of flat modules. *J. Pure Appl. Algebra*, 15(2):149–172, 1979. 5

[23] Hans-Bjørn Foxby. Homological dimensions of complexes of modules. In *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979)*, volume 795 of *Lecture Notes in Math.*, pages 360–368. Springer, Berlin, 1980. 2, 5

[24] Hans-Bjørn Foxby and Srikanth Iyengar. Depth and amplitude for unbounded complexes. In *Commutative algebra. Interactions with algebraic geometry. Proceedings of the international conference, Grenoble, France, July 9–13, 2001 and the special session at the joint international meeting of the American Mathematical Society and the Société Mathématique de France, Lyon, France, July 17–20, 2001*, pages 119–137. Providence, RI: American Mathematical Society (AMS), 2003. 5

[25] Mohsen Gheibi, David A. Jorgensen, and Ryo Takahashi. Quasi-projective dimension. *Pacific J. Math.*, 312(1):113–147, 2021. 1

[26] Craig Huneke and David A. Jorgensen. Symmetry in the vanishing of Ext over Gorenstein rings. *Math. Scand.*, 93(2):161–184, 2003. 9

[27] Craig Huneke and Roger Wiegand. Tensor products of modules and the rigidity of Tor. *Math. Ann.*, 299(3):449–476, 1994. 1

[28] Srikanth B. Iyengar. Depth for complexes, and intersection theorems. *Math. Z.*, 230(3):545–567, 1999. 1, 5, 14

[29] David A. Jorgensen. A generalization of the Auslander-Buchsbaum formula. *J. Pure Appl. Algebra*, 144(2):145–155, 1999. 2, 12

[30] David A. Jorgensen. Vanishing of (co)homology over commutative rings. *Comm. Algebra*, 29(5):1883–1898, 2001. 10, 11

[31] David A. Jorgensen. On tensor products of rings and extension conjectures. *J. Commut. Algebra*, 1(4):635–646, 2009. 9

[32] David A. Jorgensen and Liana M. Şega. Nonvanishing cohomology and classes of Gorenstein rings. *Adv. Math.*, 188(2):470–490, 2004. 8, 9, 11

[33] David A. Jorgensen and Liana M. Şega. Asymmetric complete resolutions and vanishing of Ext over Gorenstein rings. *Int. Math. Res. Not.*, (56):3459 – 3477, 2005. 9

[34] Kaito Kimura, Justin Lyle, Yuya Otake, and Ryo Takahashi. On the vanishing of Ext modules over a local unique factorization domain with an isolated singularity. *Preprint 2023; posted at arXiv:2310.16599v2*. 9

[35] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid. 7

[36] Arash Sadeghi. A note on the depth formula and vanishing of cohomology. *Preprint 2012; posted at arXiv:1204.4083v1*. 1, 10

OLGUR CELIKBAS, SCHOOL OF MATHEMATICAL AND DATA SCIENCES, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506 U.S.A

Email address: `olgur.celikbas@math.wvu.edu`

TOSHINORI KOBAYASHI, SCHOOL OF SCIENCE AND TECHNOLOGY, MEIJI UNIVERSITY, 1-1-1 HIGASHI-MITA, TAMA-KU, KAWASAKI-SHI, KANAGAWA 214-8571, JAPAN

Email address: `tkobayashi@meiji.ac.jp`

BRIAN LAVERTY, SCHOOL OF MATHEMATICAL AND DATA SCIENCES, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506 U.S.A

Email address: `bml0016@mix.wvu.edu`

HIROKI MATSUI, DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE AND TECHNOLOGY, TOKUSHIMA UNIVERSITY, 2-1 MINAMIJOSANJIMA-CHO, TOKUSHIMA 770-8506, JAPAN

Email address: `hmatsui@tokushima-u.ac.jp`