PRETHICK SUBCATEGORIES OF MODULES AND CHARACTERIZATIONS OF LOCAL RINGS

HIROKI MATSUI AND HAYATO MURATA

Abstract. "This paper studies characterizing local rings in terms of homological dimensions. The key tool is the notion of a prethick subcategory which we introduce in this paper. Our methods recover the theorems of Salarian, Sather-Wagstaff and Yassemi.

1. Introduction

Throughout this paper, let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules.

We call a full subcategory of $\operatorname{mod} R$ *prethick* if it is closed under finite direct sums, direct summands, kernels of epimorphisms and cokernels of monomorphisms. A prethick subcategory closed under extensions is often called a *thick* subcategory, which has been well investigated so far; see [3, 6, 9]. What we want to study in this paper is the following question.

Question 1.1. When does a prethick subcategory of $\operatorname{mod} R$ contain the residue field k?

The main results of this paper are the following two theorems.

Theorem 1.2. A prethick subcategory $\mathfrak{X}$ of $\operatorname{mod} R$ contains k if there exists a finitely generated R-module M satisfying the following two conditions:

(1) $\operatorname{depth} R \geq \operatorname{depth}_R M + 1$.
(2) $M/(x_1, \ldots, x_i)M$ is in $\mathfrak{X}$ for all $i = 0, \ldots, \operatorname{depth}_R M+1$ and all R-regular sequences $x_1, \ldots, x_i$.

Theorem 1.3. A prethick subcategory $\mathfrak{X}$ of $\operatorname{mod} R$ contains k if there exists a finitely generated R-module M satisfying the following two conditions:

(1) $\dim R \geq \operatorname{depth}_R M + 1$.
(2) $M/(x_1, \ldots, x_i)M$ is in $\mathfrak{X}$ for all $i = 0, \ldots, \operatorname{depth}_R M + 1$ and all subsystems of parameters $x_1, \ldots, x_i$ for R.

Using these theorems, we can recover all the results given in [7], and furthermore yield new results on Tor and Ext modules, complexities and Betti numbers. In Section 2 we give the precise definition of a prethick subcategory and make some examples. In Section 3 we prove the above two theorems and apply them to recover the results in [7] and obtain new results.

Date: April 30, 2018.
2010 *Mathematics Subject Classification.* 13C60, 13D05, 13H10.
Key words and phrases. homological dimension, (pre)thick subcategory, regular sequence, system of parameters.

2. Prethick subcategories

In this section we define a prethick subcategory and give some examples.

Definition 2.1. A full subcategory $\mathfrak{X}$ of $\mathsf{mod}\, R$ is said to be *prethick* if $\mathfrak{X}$ satisfies the following conditions.

(1) $\mathfrak{X}$ is closed under isomorphisms: if M is in $\mathfrak{X}$ and $N \in \mathsf{mod}\, R$ is isomorphic to M, then N is also in $\mathfrak{X}$.
(2) $\mathfrak{X}$ is closed under finite direct sums: if $M_1, \dots, M_n$ are in $\mathfrak{X}$, so is the direct sum $M_1 \oplus \cdots \oplus M_n$.
(3) $\mathfrak{X}$ is closed under direct summands: if M is in $\mathfrak{X}$ and N is a direct summand of M, then N is also in $\mathfrak{X}$.
(4) $\mathfrak{X}$ is closed under kernels of epimorphisms: for any exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$, if M and N are in $\mathfrak{X}$, then so is L.
(5) $\mathfrak{X}$ is closed under cokernels of monomorphisms: for any exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$, if L and M are in $\mathfrak{X}$, then so is N.

Example 2.2. Let N be an R-module, C a semidualizing R-module and I an ideal of R. The full subcategory of $\mathsf{mod}\, R$ consisting of modules X satisfying the property $\mathbb{P}$ is prethick, where $\mathbb{P}$ is one of the following:

(1) $\mathrm{G}_C\text{-}\dim_R X < \infty$.	(2) $\mathrm{G}\text{-}\dim_R X < \infty$.	(3) $\mathrm{CI}_*\text{-}\dim_R X < \infty$.
(4) $\mathrm{Gid}_R X < \infty$.	(5) $\mathrm{pd}_R X < \infty$.	(6) $\mathrm{id}_R X < \infty$.
(7) $\mathrm{Tor}^R_{\gg 0}(X, N) = 0$.	(8) $\mathrm{Ext}_R^{\gg 0}(X, N) = 0$.	(9) $\mathrm{Ext}_R^{\gg 0}(N, X) = 0$.
(10) $\mathrm{cx}_R X < \infty$.	(11) $\sup_{i \geq 0}\{\beta_i^R(X)\} < \infty$.	(12) $IX = 0$.

Here, G_C-dim, G-dim, CI_*-dim, Gid, cx and β_i denote Gorenstein dimension with respect to C, Gorenstein dimension, lower complete intersection dimension, Gorenstein injective dimension, complexity and i-th Betti number, respectively. For their definitions, we refer the reader to [2, 7].

Proof. We give a proof of the assertion only for the property (1). The assertion for the other properties is shown easily, or similarly, or by [2, Theorem 4.2.4], [4, Theorem 2.25], [8, Example 2.4 (10)].

Set $n = \operatorname{depth} R$. Let $\mathfrak{X}$ be the full subcategory of $\mathsf{mod}\, R$ consisting of all modules X with $\mathrm{G}_C\text{-}\dim_R X < \infty$. It is easy to see from the definition of G_C-dim that $\mathfrak{X}$ is closed under isomorphisms, finite direct sums and direct summands. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence in $\mathsf{mod}\, R$ with $\mathrm{G}_C\text{-}\dim_R M < \infty$. Taking the d-th syzygies, where $d = \dim R$, induces an exact sequence

$$0 \to \Omega^d L \to \Omega^d M \xrightarrow{\alpha} \Omega^d N \to 0$$

up to free summands. Note have that $\Omega^d M$ is totally C-reflexive. It follows from [1, Theorem 2.1] that if $\Omega^d N$ is totally C-reflexive, then so is $\Omega^d L$. This shows that $\mathfrak{X}$ is closed under kernels of epimorphisms. Take an exact sequence

$$0 \to \Omega^{d+1} N \to F \xrightarrow{\beta} \Omega^d N \to 0$$

with F free. The pullback diagram of α and β yields an exact sequence

$$0 \to \Omega^{d+1} N \to \Omega^d L \oplus F \to \Omega^d M \to 0.$$

Again by [1, Theorem 2.1] we observe that if $\Omega^d L$ is totally C-reflexive, then so is $\Omega^{d+1} N$. This implies that $\mathfrak{X}$ is closed under cokernels of monomorphisms. $\square$

Remark 2.3. A full subcategory $\mathfrak{X}$ of $\operatorname{mod} R$ is said to be *closed under extensions* provided that for an exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if L and N are in $\mathfrak{X}$, then so is M. A prethick subcategory closed under extensions is called a *thick* subcategory. By definition any thick subcategory is prethick, but the converse does not necessarily hold. In fact, let $\mathfrak{X}$ be the full subcategory of $\operatorname{mod} R$ consisting of all modules that are annihilated by the maximal ideal $\mathfrak{m}$. As we saw in Example 2.2, $\mathfrak{X}$ is a prethick subcategory of $\operatorname{mod} R$. Suppose that $\mathfrak{X}$ is closed under extensions. Then consider the natural short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}^2 \to k \to 0$$

in $\operatorname{mod} R$. Since k and $\mathfrak{m}/\mathfrak{m}^2$ belong to $\mathfrak{X}$ and $\mathfrak{X}$ is assumed to be closed under extensions, $R/\mathfrak{m}^2$ also belongs to $\mathfrak{X}$, which means $\mathfrak{m} = \mathfrak{m}^2$. Nakayama's lemma implies $\mathfrak{m} = 0$, and it follows that R is a field. Consequently, unless R is a field, the subcategory $\mathfrak{X}$ is a prethick non-thick subcategory.

3. Main results and their applications

In this section, we prove our main results and provide several results as corollaries, including the results in [7]. Let us state and prove our first main result.

Theorem 3.1. Let M be a finitely generated R-module and $\mathfrak{X}$ a prethick subcategory of $\operatorname{mod} R$. If M satisfies the following conditions, then $\mathfrak{X}$ contains k.

(1) $\operatorname{depth} R \geq \operatorname{depth}_R M + 1$.
(2) $M/(x_1, \dots, x_i)M$ is in $\mathfrak{X}$ for all $i = 0, \dots, \operatorname{depth}_R M + 1$ and all R-regular sequences $x_1, \dots, x_i$.

Proof. We use induction on $\operatorname{depth}_R M$. First, consider the case $\operatorname{depth}_R M = 0$. By assumption we have $\operatorname{depth} R > 0$. Therefore, by prime avoidance, we can take an R-regular element $x \in \mathfrak{m}$ such that

$$x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass} M \setminus \{\mathfrak{m}\}} \mathfrak{p}.$$

Since M is Noetherian, there exists an integer $\alpha \geq 0$ such that $[0 :_M x^\alpha] = [0 :_M x^{\alpha+i}]$ for all $i \geq 0$. The assumption $\operatorname{depth}_R M = 0$ yields $[0 :_M x^\alpha] \neq 0$, and set $N = [0 :_M x^\alpha]$. The exact sequence

$$0 \longrightarrow N \longrightarrow M \xrightarrow{x^\alpha} M \longrightarrow M/x^\alpha M \longrightarrow 0$$

induces $N \in \mathfrak{X}$. Since $\operatorname{Ass} N = \operatorname{Ass} \operatorname{Hom}(R/(x^\alpha), M) = V(x) \cap \operatorname{Ass} M = \{\mathfrak{m}\}$, the R-module N has finite length. Hence there exists an integer $n > 0$ such that $\mathfrak{m}^n N = 0$ and $\mathfrak{m}^{n-1} N \neq 0$. Note that $N = \Gamma_\mathfrak{m}(M)$. Since $\operatorname{depth}_R M/N > 0$, we can take an M/N-regular and R-regular element $y \in \mathfrak{m}^{n-1}$ such that

$$y \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass} M \setminus \{\mathfrak{m}\}} \mathfrak{p} \cup \operatorname{Ann} N.$$

As M is noetherian, there exists an integer $\beta \geq 0$ such that $[0 :_M y^\beta] = [0 :_M y^{\beta+i}]$ for all $i \geq 0$. The above argument implies $[0 :_M y^\beta] = \Gamma_\mathfrak{m}(M) = N$. From the short exact sequence

$$(*) \qquad 0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

we have $M/N \in \mathfrak{X}$. The fact that y is M/N-regular gives two exact sequences

$$0 \longrightarrow M/N \xrightarrow{y} M/N \longrightarrow M/(yM+N) \longrightarrow 0,$$

$$0 \longrightarrow N/yN \longrightarrow M/yM \longrightarrow M/(yM+N) \longrightarrow 0,$$

where the latter one is induced by applying $R/(y) \otimes_R -$ to $(*)$. It is seen that $M/(yM+N)$ belongs to $\mathfrak{X}$, and hence so does N/yN. The natural exact sequence

$$0 \longrightarrow yN \longrightarrow N \longrightarrow N/yN \longrightarrow 0$$

shows $yN \in \mathfrak{X}$. As $yN \neq 0$ and $\mathfrak{m}(yN) \subseteq \mathfrak{m}^n N = 0$, we see that yN is a direct sum of k. Since $\mathfrak{X}$ is closed under direct summands, $\mathfrak{X}$ contains k.

Next, let us consider the case $\operatorname{depth}_R M > 0$. By prime avoidance, we can take an R-regular and M-regular element $x \in \mathfrak{m}$. Set $\overline{R} = R/(x)$ and $\overline{M} = M/xM$. Fix an integer $0 \leq i \leq \operatorname{depth}_R \overline{M} + 1$ and take an $\overline{R}$-regular sequence $\overline{x_1}, \dots, \overline{x_i}$. Then $0 \leq 1 \leq i+1 \leq \operatorname{depth}_R M + 1$, and $x, x_1, \dots, x_i$ is an R-regular sequence. Note that $\overline{M}/(\overline{x_1}, \dots, \overline{x_i})\overline{M} = M/(x, x_1, \dots, x_i)M$ is in $\mathfrak{X}$. Let $\overline{\mathfrak{X}}$ be the full subcategory of $\operatorname{\mathsf{mod}} \overline{R}$ consisting of all $\overline{R}$-modules that belong to $\mathfrak{X}$ as R-modules. Then it is easy to see that $\overline{\mathfrak{X}}$ is a prethick subcategory of $\operatorname{\mathsf{mod}} \overline{R}$, and the $\overline{R}$-module $\overline{M}/(\overline{x_1}, \dots, \overline{x_i})\overline{M}$ belongs to $\overline{\mathfrak{X}}$. The induction hypothesis implies that k is in $\overline{\mathfrak{X}}$, and so we get $k \in \mathfrak{X}$. □

The replacement

$\operatorname{depth} R$	$\longmapsto$	$\dim R$,
$\operatorname{Ass} R$	$\longmapsto$	$\operatorname{Min} R$ (or $\operatorname{Assh} R$),
R- regular sequence	$\longmapsto$	subsystem of parameters for R,
$\overline{R}$- regular sequence	$\longmapsto$	subsystem of parameters for $\overline{R}$

in the proof of Theorem 3.1 yields our second main result:

Theorem 3.2. Let M be a finitely generated R-module and $\mathfrak{X}$ a prethick subcategory of $\operatorname{\mathsf{mod}} R$. If M satisfies the following conditions, then $\mathfrak{X}$ contains k.

(1) $\dim R \geq \operatorname{depth}_R M + 1$.
(2) $M/(x_1, \dots, x_i)M$ is in $\mathfrak{X}$ for all $i = 0, \dots, \operatorname{depth}_R M + 1$ and all subsystems of parameters $x_1, \dots, x_i$ for R.

Our Theorem 3.1 and 3.2 recover all the results shown in [7]. In what follows, let Φ be any of the homological dimensions $\mathrm{G}_C\text{-}\dim_R$, $\mathrm{G}\text{-}\dim_R$, $\mathrm{CI}_*\text{-}\dim_R$, Gid_R, pd_R, and id_R.

Corollary 3.3. [7, Theorem 3 and Corollaries 4–9] Let M be a finitely generated R-module and $0 \leq t \leq d := \operatorname{depth} R$ an integer. If $\Phi(M/(x_1, \dots, x_i)M) < \infty$ for all $0 \leq i \leq d-t$ and all R-regular sequences $x_1, \dots, x_i$, then one has either $\operatorname{depth}_R M \geq d-t$ or $\Phi(k) < \infty$.

Proof. By Example 2.2, the finitely generated R-modules X with $\Phi(X) < \infty$ form a prethick subcategory of $\operatorname{\mathsf{mod}} R$. If $\operatorname{depth}_R M \leq d-t-1$, then we have $\operatorname{depth} R \geq \operatorname{depth}_R M + t + 1 \geq \operatorname{depth}_R M + 1 \leq d - t$. Hence $\Phi(k)$ is finite by Theorem 3.1. □

Remark 3.4. In Corollary 3.3, the inequality $\operatorname{depth}_R M \geq d-t$ is equivalent to the inequality $\Phi(M) \leq t$ in the case where Φ is one of the homological dimensions $\mathrm{G}_C\text{-}\dim_R$, $\mathrm{G}\text{-}\dim_R$, $\mathrm{CI}_*\text{-}\dim_R$ and pd_R, because such Φ satisfies an Auslander-Buchsbaum-type equality.

Corollary 3.5. [7, Corollary 10] Let M be a finitely generated R-module. The following conditions are equivalent.

(1) R is Cohen-Macaulay.
(2) There exists a finitely generated R-module M such that $\mathrm{G}_C\text{-}\dim_R(M/\mathfrak{a}M)$ is finite for every ideal $\mathfrak{a}$ generated by a subsystem of parameters for R.
(3) For every ideal $\mathfrak{a}$ generated by a subsystem of parameters for R, one has $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{a})$ is finite.

Proof. (1) $\Longrightarrow$ (3): Since R is Cohen-Macaulay, $\mathfrak{a}$ is generated by an R-regular sequence, so $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{a})$ is finite.
(3) $\Longrightarrow$ (2): This implication is shown by letting $M = R$.
(2) $\Longrightarrow$ (1): Assume that R is not Cohen-Macaulay, then we have $\operatorname{depth}_R M \leq \operatorname{depth} R < \dim R$. From Theorem 3.2, we get $\mathrm{G}_C\text{-}\dim_R k < \infty$ and it follows that R is Cohen-Macaulay. This is a contradiction. □

In relation to Corollary 3.5, one can also deduce the result below from Theorem 3.2.

Corollary 3.6. Let $\Phi \in \{\mathrm{G}_C\text{-}\dim_R, \mathrm{G}\text{-}\dim_R, \mathrm{CI}_*\text{-}\dim_R, \mathrm{pd}_R\}$. If there exists a finitely generated R-module M such that $0 < \Phi(M/\underline{x}M) < \infty$ for all subsystems of parameters $\underline{x}$ for R, then $\Phi(k) < \infty$.

Proof. As $0 < \Phi(M) < \infty$, we have $\operatorname{depth} R - \operatorname{depth}_R M = \Phi(M) \geq 1$, so $\dim R \geq \operatorname{depth}_R M + 1$. Theorem 3.2 implies $\Phi(k) < \infty$. □

Using Theorem 3.1 and Example 2.2, we obtain the following new results concerning Tor and Ext modules, complexities and Betti numbers.

Corollary 3.7. Let M, N be finitely generated R-modules with $\operatorname{depth} R > \operatorname{depth}_R M$. If $\operatorname{Tor}^R_{\gg 0}(M/(x_1,\dots,x_i)M, N) = 0$ (respectively, $\operatorname{Ext}_R^{\gg 0}(M/(x_1,\dots,x_i)M, N) = 0$) for all $i = 0,\dots,\operatorname{depth}_R M + 1$ and all R-regular sequences $x_1,\dots,x_i$, then $\mathrm{pd}_R N < \infty$ (respectively, $\mathrm{id}_R N < \infty$).

Corollary 3.8. Let M be a finitely generated R-module with $\operatorname{depth} R > \operatorname{depth}_R M$. If the R-module $M/(x_1,\dots,x_i)M$ has finite complexity (respectively, bounded Betti numbers) for all $i = 0,\dots,\operatorname{depth}_R M+1$ and all R-regular sequences $x_1,\dots,x_i$, then R is a complete intersection (respectively, hypersurface).

For the proof of Corollary 3.8 we use the fact that a local ring R whose residue field has finite complexity (respectively, bounded Betti numbers) as an R-module is a complete intersection (respectively, hypersurface).

Acknowledgments

The authors would like to express their deep gratitude to their supervisor Ryo Takahashi for a lot of comments, suggestions and discussions.

References

[1] T. Araya, R. Takahashi and Y. Yoshino, Homological invariants associated to semi-dualizing bimodules, *J. Math. Kyoto Univ.* **45** (2005), 287–306.
[2] L.L. Avramov, Infinite free resolutions. Six lectures on commutative algebra, *Progr. Math.* **166** (1998), 1–118.
[3] D. Benson, S. B. Iyengar and H. Krause, Stratifying modular representations of finite groups, *Ann. of Math.* **174** (2011), no. 3 1643–1684. @
[4] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra.* **189** (2004), 167–193.
[5] H. Krause, Thick subcategories of modules over commutative Noetherian ring (with an appendix by Srikanth Iyengar). *Math. Ann.* **340** (2008), 733–747.

[6] H. Krause and G. Stevenson, A note on thick subcategories of stable derived categories. *Nagoya Math. J.* **212** (2013), 87–96.

[7] S. Salarian, S. Sather-Wagstaff and S. Yassemi, Characterizing local rings via homological dimensions and regular sequences, *J. Pure Appl. Algebra.* **207** (2006), 99–108.

[8] R. Takahashi, Modules in resolving subcategories which are free on the punctured spectrum, *Pacific J. Math.* **241** (2009), 347–367.

[9] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan.* **65** (2013), no. 2, 357–374.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

Email address: m14037f@math.nagoya-u.ac.jp

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

Email address: m13047e@math.nagoya-u.ac.jp