CLASSIFYING DENSE RESOLVING AND CORESOLVING SUBCATEGORIES OF EXACT CATEGORIES VIA GROTHENDIECK GROUPS

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ABSTRACT. Classification problems of subcategories have been deeply considered so far. In this paper, we discuss classifying dense resolving and dense coresolving subcategories of exact categories via their Grothendieck groups. This study is motivated by the classification of dense triangulated subcategories of triangulated categories due to Thomason.

1. INTRODUCTION

Let $\mathcal{C}$ be a category. Classifying subcategories means for a property $\mathbb{P}$, finding a one-to-one correspondence

$$\{\text{subcategories of } \mathcal{C} \text{ satisgying } \mathbb{P}\} \underset{g}{\overset{f}{\rightleftarrows}} S,$$

where the set S is easier to understand. Classifying subcategories is an important approach to understand the category $\mathcal{C}$ and has been studied in various areas of mathematics, for example, stable homotopy theory, commutative/noncommutative ring theory, algebraic geometry, and modular representation theory.

Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ a full additive subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ is *additively closed* if it is closed under taking direct summands, and that $\mathcal{X}$ is *dense* if any object in $\mathcal{A}$ is a direct summand of some object of $\mathcal{X}$. We can easily show that $\mathcal{X}$ is additively closed if and only if $\mathcal{X} = \mathsf{add}\,\mathcal{X}$ and that $\mathcal{X}$ is dense if and only if $\mathcal{A} = \mathsf{add}\,\mathcal{X}$. Here, $\mathsf{add}\,\mathcal{X}$ denotes the smallest full additive subcategory of $\mathcal{A}$ which is closed under taking direct summands and contains $\mathcal{X}$. Therefore, for any full additive subcategory $\mathcal{X}$ of $\mathcal{A}$, $\mathcal{X}$ is a dense subcategory of $\mathsf{add}\,\mathcal{X}$ and $\mathsf{add}\,\mathcal{X}$ is an additively closed subcategory of $\mathcal{A}$. For this reason, to classify additive subcategories, it suffices to classify additively closed ones and dense ones. Classification of additively closed or dense subcategories has been deeply studied so far. For example, the following three kinds of subcategories have been classified by Gabriel [10], Hopkins and Neeman [13, 18], and Thomason [23], respectively.

(1) The Serre subcategories of finitely generated modules over a commutative noetherian ring.

2010 *Mathematics Subject Classification.* 18E10, 18F30, 16G50.

Key words and phrases. exact category, dense subcategory, resolving subcategory, coresolving subcategory, Grothendieck group.

The author is supported by Grant-in-Aid for JSPS Fellows 16J01067.

(2) The thick subcategories of perfect complexes over a commutative noetherian ring.
(3) The dense triangulated subcategories of an essentially small triangulated category.

(1) and (2) are classifications of additively closed subcategories, while (3) is a classification of dense subcategories.

Let us state the precise statement of Thomason's classification theorem.

Theorem 1.1 (Thomason)**.** *Let $\mathcal{T}$ be an essentially small triangulated category. Then there is a one-to-one correspondence*

$$\{\textit{dense triangulated subcategories of } \mathcal{T}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{subgroups of } K_0(\mathcal{T})\},$$

where f and g are given by $f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X}\rangle$ and $g(H) := \{X \in \mathcal{T} \mid [X] \in H\}$, respectively, and $K_0(\mathcal{T})$ stands for the Grothendieck group of $\mathcal{T}$.

Motivated by this theorem, we discuss classifying dense resolving and dense coresolving subcategories of exact categories. The notion of a resolving subcategory has been introduced by Auslander and Bridger [2] and that of a coresolving subcategory is its dual notion. Resolving and coresolving subcategories have been widely studied so far, for example, see [1, 3, 8, 14, 21, 22]. The main theorem of this paper is the following.

Theorem 1.2 (Proposition 2.5, Theorem 2.7)**.** *Let $\mathcal{E}$ be an essentially small exact category with either a generator or a cogenerator $\mathcal{G}$.*

(1) *The following subcategories of $\mathcal{E}$ are the same:*
   (i) *dense $\mathcal{G}$-resolving subcategories*
   (ii) *dense $\mathcal{G}$-coresolving subcategories*
   (iii) *dense $\mathcal{G}$-2-out-of-3 subcategories*
(2) *There is a one-to-one correspondence*

$$\{\textit{dense } \mathcal{G}\textit{-(co)resolving subcategories of } \mathcal{E}\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\textit{subgroups of } K_0(\mathcal{E}) \\ \textit{containing the image of } \mathcal{G}\end{matrix}\right\},$$

where f and g are given by $f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X}\rangle$ and $g(H) := \{X \in \mathcal{E} \mid [X] \in H\}$, respectively, and $K_0(\mathcal{E})$ stands for the Grothendieck group of $\mathcal{E}$.

Here, the notion of a $\mathcal{G}$-resolving (resp. $\mathcal{G}$-coresolving) subcategory is a slight generalization of that of a resolving (resp. coresolving) subcategory. Indeed, they coincide when $\mathcal{G}$ consists of the projective (resp. injective) objects. In addition, $\mathcal{G}$-2-out-of-3 subcategory is a subcategory which is both $\mathcal{G}$-resolving and $\mathcal{G}$-coresolving. The precise definitions of these subcategories will be given in Definition 2.3.

The organization of this paper is as follows. In Section 2, we give a proof of our main theorem and several corollaries which include a correspondence between dense resolving and dense coresolving subcategories of an exact category and dense triangulated subcategories of its derived category. In Section 3, we discuss relation between dense subcategories of exact categories and that of triangulated categories.

In Section 4, as applications of our results, we discuss when there are only finitely many dense resolving subcategories of finitely generated modules over a left noetherian ring.

2. Classification of dense resolving subcategories

In this section, we show our main result. Throughout this paper, let $\mathcal{A}$ be an abelian category, $\mathcal{E}$ an exact category, and $\mathcal{T}$ a triangulated category. We always assume that all categories are essentially small, and that all subcategories are full and additive. For a left noetherian ring A, $\mathsf{mod}\, A$ denotes the category of finitely generated left A-modules.

We begin with recalling several notions, which are key notions of this paper.

Definition 2.1. Let $\mathcal{G}$ be a family of objects of $\mathcal{E}$. We call $\mathcal{G}$ a *generator* (resp. a *cogenerator*) of $\mathcal{E}$ if for any object $A \in \mathcal{E}$, there is a short exact sequence

$$A' \rightarrowtail G \twoheadrightarrow A \ \ (\text{resp. } A \rightarrowtail G \twoheadrightarrow A')$$

in $\mathcal{E}$ with $G \in \mathcal{G}$.

Example 2.2. (1) Clearly, $\mathcal{E}$ is both a generator and a cogenerator of $\mathcal{E}$

(2) If $\mathcal{E}$ has enough projective (resp. injective) objects, then the subcategory $\mathsf{proj}\, \mathcal{E}$ (resp. $\mathsf{inj}\, \mathcal{E}$) consisting of projective (resp. injective) objects is a generator (resp. a cogenerator) of $\mathcal{E}$.

Next we give the definitions of $\mathcal{G}$-resolving and $\mathcal{G}$-coresolving subcategories.

Definition 2.3. Let $\mathcal{X}$ be a subcategory of $\mathcal{E}$ and $\mathcal{G}$ a family of objects of $\mathcal{E}$.

(1) We say that $\mathcal{X}$ is a $\mathcal{G}$-*resolving subcategory* of $\mathcal{E}$ if the following three conditions are satisfied:
   - (i) $\mathcal{X}$ is closed under extensions: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if X and Z are in $\mathcal{X}$, then so is Y.
   - (ii) $\mathcal{X}$ is closed under kernels of admissible epimorphisms: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if Y and Z are in $\mathcal{X}$, then so is X.
   - (iii) $\mathcal{X}$ contains $\mathcal{G}$.

   If $\mathcal{E}$ has enough projective objects, we shall call $\mathcal{X}$ simply *resolving* if it is $\mathsf{proj}\, \mathcal{E}$-resolving.

(2) We say that $\mathcal{X}$ is a $\mathcal{G}$-*coresolving subcategory* of $\mathcal{E}$ if the following three conditions are satisfied:
   - (i) $\mathcal{X}$ is closed under extensions: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if X and Z are in $\mathcal{X}$, then so is Y.
   - (ii) $\mathcal{X}$ is closed under cokernels of admissible monomorphisms: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if X and Y are in $\mathcal{X}$, then so is Z.
   - (iii) $\mathcal{X}$ contains $\mathcal{G}$.

(3) We say that $\mathcal{X}$ is a $\mathcal{G}$-*2-out-of-3 subcategory* of $\mathcal{E}$ if the following conditions are satisfied:
   - (i) $\mathcal{X}$ satisfies 2-out-of-3 property: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if 2 out of $\{X, Y, Z\}$ belong to $\mathcal{X}$, then so is the third.
   - (ii) $\mathcal{X}$ contains $\mathcal{G}$.

Remark 2.4. Unlike the definition due to Auslander and Bridger [2], we do not assume that resolving subcategories are closed under direct summands. Therefore, our definition is rather close to the definitions in [3, 14].

The following proposition shows that dense $\mathcal{G}$-resolving, dense $\mathcal{G}$-coresolving, and dense $\mathcal{G}$-2-out-of-3 subcategories are the same thing.

Proposition 2.5. *Let $\mathcal{X}$ be a dense subcategory of $\mathcal{E}$.Then $\mathcal{X}$ is closed under cokernels of admissible monomorphisms if and only if it is closed under kernels of admissible epimorphisms.*

Proof. We have only to show the 'if' part. The 'only if' part is proved by the dual argument.

Let $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ be a short exact sequence in $\mathcal{E}$ with $X, Y \in \mathcal{X}$. Since $\mathcal{X}$ is dense, we can take $Z' \in \mathcal{E}$ with $Z \oplus Z' \in \mathcal{X}$. Consider a short exact sequence

$$X \oplus Z \overset{\left(\begin{smallmatrix} f & 0 \\ 0 & id_Z \\ 0 & 0 \end{smallmatrix}\right)}{\rightarrowtail} Y \oplus Z \oplus Z' \overset{\left(\begin{smallmatrix} g & 0 & 0 \\ 0 & 0 & id_{Z'} \end{smallmatrix}\right)}{\twoheadrightarrow} Z \oplus Z'.$$

Then $X \oplus Z$ is an object of $\mathcal{X}$ because $\mathcal{X}$ is closed under kernels of admissible epimorphisms. From the split short exact sequence $Z \rightarrowtail X \oplus Z \twoheadrightarrow X$, we obtain $Z \in \mathcal{X}$ since $\mathcal{X}$ is closed under kernels of admissible epimorphisms. ■

Now we recall the definition of the Grothendieck group of an exact category.

Definition 2.6. Let $\mathcal{E}$ be an exact category. Let F be the free abelian group generated by the isomorphism classes of objects of $\mathcal{E}$. Let I be the subgroup of F generated by the elements of the form $[A] - [B] + [C]$ where $A \rightarrowtail B \twoheadrightarrow C$ are short exact sequences in $\mathcal{E}$. Then we define the *Grothendieck group* of $\mathcal{E}$, denoted by $K_0(\mathcal{E})$, as the quotient group F/I.

The following theorem is our main result of this paper.

Theorem 2.7. *Let $\mathcal{E}$ be an essentially small exact category with either a generator or a cogenerator $\mathcal{G}$. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{E}$*$\}$,
(2) $\{$*dense $\mathcal{G}$-coresolving subcategories of $\mathcal{E}$*$\}$,
(3) $\{$*dense $\mathcal{G}$-2-out-of-3 subcategories of $\mathcal{E}$*$\}$, *and*
(4) $\{$*subgroups of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$*$\}$.

One-to-one correspondences among (1), (2) and (3) have been already shown in Proposition 2.5. Thus, it suffice to show the bijection between (1) and (4). Moreover, we will show this bijection only in the case that $\mathcal{G}$ is a generator because in the cogenerator case, it can be shown by the dual argument. The following lemma is essential in the proof of our theorem.

Lemma 2.8. *Let $\mathcal{G}$ be a generator of $\mathcal{E}$ and $\mathcal{X}$ a dense $\mathcal{G}$-resolving subcategory of $\mathcal{E}$. Then for an object A in $\mathcal{E}$, $A \in \mathcal{X}$ if and only if $[A] \in \langle [X] \mid X \in \mathcal{X} \rangle$.*

Proof. Define an equivalence relation $\sim$ on the isomorphism classes $\mathcal{E}/_{\cong}$ of objects of $\mathcal{E}$, as follows: $A \sim A'$ if there are $X, X' \in \mathcal{X}$ such that $A \oplus X \cong A' \oplus X'$. Set $\langle \mathcal{E} \rangle_{\mathcal{X}} := (\mathcal{E}/_{\cong})/_{\sim}$ and denote by $\langle A \rangle$ the class of A. Then $\langle \mathcal{E} \rangle_{\mathcal{X}}$ is an abelian group with $\langle A \rangle + \langle B \rangle := \langle A \oplus B \rangle$. Indeed, obviously, $+$ is well-defined, commutative, associative, and $\langle 0 \rangle$ is an identity element. Since $\mathcal{X}$ is dense, for any $A \in \mathcal{E}$, there is $A' \in \mathcal{E}$ such that $A \oplus A' \in \mathcal{X}$, and hence $\langle A \rangle + \langle A' \rangle = \langle A \oplus A' \rangle = \langle 0 \rangle$. Therefore, $\langle A' \rangle$ is an inverse element of $\langle A \rangle$.

Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ be a short exact sequence in $\mathcal{E}$. Taking $A', C' \in \mathcal{E}$ with $A \oplus A', C \oplus C' \in \mathcal{X}$ and considering a short exact sequence

$$A \oplus A' \overset{\begin{pmatrix} f & 0 \\ 0 & id_{A'} \\ 0 & 0 \end{pmatrix}}{\rightarrowtail} B \oplus A' \oplus C' \overset{\begin{pmatrix} g & 0 & 0 \\ 0 & 0 & id_{C'} \end{pmatrix}}{\twoheadrightarrow} C \oplus C'.$$

we have $B \oplus A' \oplus C' \in \mathcal{X}$. This shows $\langle B \rangle - \langle A \rangle - \langle C \rangle = \langle B \oplus A' \oplus C' \rangle = \langle 0 \rangle$. Therefore, there is a group homomorphism

$$\varphi : K_0(\mathcal{E}) \to \langle \mathcal{E} \rangle_{\mathcal{X}}, \quad [A] \mapsto \langle A \rangle.$$

Note that $\langle [X] \mid X \in \mathcal{X} \rangle$ is contained in $\mathsf{Ker}\, \varphi$.

From the definition of the Grothendieck group, any element of $K_0(\mathcal{E})$ is denoted by $[A] - [B]$. Moreover, since there is a short exact sequence

$$B' \rightarrowtail G \twoheadrightarrow B$$

in $\mathcal{E}$ with $G \in \mathcal{G}$, $[A] - [B] = [A \oplus B'] - [G]$. Thus, any element of $K_0(\mathcal{E})$ is denoted by $[A] - [G]$ with $G \in \mathcal{G}$.

Let $[A] - [G]$ with $G \in \mathcal{G}$ be an element of $\mathsf{Ker}\, \varphi$. Since $\mathcal{X}$ contains $\mathcal{G}$, $[A] \in \mathsf{Ker}\, \varphi$. This means $\langle A \rangle = \langle 0 \rangle$ and there are $X, X' \in \mathcal{X}$ such that $A \oplus X \cong X'$. Considering the split short exact sequence

$$A \rightarrowtail A \oplus X \twoheadrightarrow X,$$

we obtain $A \in \mathcal{X}$ since $\mathcal{X}$ is closed under kernels of epimorphisms. Thus, $A \in \mathcal{X}$ if and only if $[A] \in \langle [X] \mid X \in \mathcal{X} \rangle$. ■

Proof of Theorem 2.7. By Lemma 2.5, the set (2) is nothing but the set (1). Therefore, we show that there is a one-to-one correspondence between the sets (1) and (3).

For a dense $\mathcal{G}$-resolving subcategory $\mathcal{X}$, define

$$f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X} \rangle,$$

and for a subgroup H of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$, define

$$g(H) := \{ A \in \mathcal{E} \mid [A] \in H \}.$$

We show that f and g give mutually inverse bijections between (1) and (3).

First note that $g(H) := \{ A \in \mathcal{E} \mid [A] \in H \}$ is a dense $\mathcal{G}$-resolving subcategory of $\mathcal{E}$ for a subgroup H of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$. Indeed, for any object $A \in \mathcal{E}$, take a short exact sequence $A' \rightarrowtail G \twoheadrightarrow A$ in $\mathcal{E}$ with $G \in \mathcal{G}$. Then $[A \oplus A'] = [A] + [A'] = [G] \in H$, and hence $A \oplus A' \in g(H)$. Thus $g(H)$ is dense. Obviously, $g(H)$ contains $\mathcal{G}$. Furthermore, for any short exact sequence

$A \rightarrowtail B \twoheadrightarrow C$, the relation $[A]-[B]+[C]=0$ implies that $g(H)$ is $\mathcal{G}$-resolving. Besides, $f(\mathcal{X})$ is clearly a subgroup of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$. As a result, f and g are well-defined maps between the sets (1) and (3)

Let H be a subgroup of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$. Then the inclusion $fg(H) \subset H$ is trivial. For any $[A]-[G] \in H$ with $G \in \mathcal{G}$, $[A]=([A]-[G])+[G] \in H$ implies $A \in g(H)$, and thus $[A]-[G] \in fg(H)$. Therefore, $fg(H)=H$.

Let $\mathcal{X}$ be a dense resolving subcategory of $\mathcal{E}$ containing $\mathcal{G}$. Then the inclusion $\mathcal{X} \subset gf(\mathcal{X})$ is trivial. Conversely, for any $A \in gf(\mathcal{X})$, since $[A] \in f(\mathcal{X}) = \langle [X] \mid X \in \mathcal{X}\rangle$, we have $A \in \mathcal{X}$ by Lemma 2.8. Therefore, $gf(\mathcal{X})=\mathcal{X}$. Consequently, f and g are mutually inverse bijections between (1) and (3). ■

3. Relations with dense triangulated subcategories

In this section, we consider some combinations of Theorem 1.1 and Theorem 2.7. Let us start with the definition of the Grothendieck group for a triangulated category.

Definition 3.1. Let $\mathcal{T}$ be a triangulated category. Let F be the free abelian group generated by the isomorphism classes of objects of $\mathcal{T}$. Let I be the subgroup generated by the elements of the form $[A]-[B]+[C]$ where $A \to B \to C \to A[1]$ are exact triangles in $\mathcal{T}$. Then we define the *Grothendieck group* of $\mathcal{T}$, denoted by $K_0(\mathcal{T})$, as the quotient group F/I.

First, we discuss dense subcategories of exact categories and their derived categories. Please refer to [7, 17] for the definition of the derived category of an exact category.

Lemma 3.2. [24, Lemma 9.2.4] *Let $\mathcal{E}$ be an essentially small exact category. Then the canonical functor $\mathcal{E} \to \mathsf{D}^{\mathsf{b}}(\mathcal{E})$ induces an isomorphism $\varphi : K_0(\mathcal{E}) \to K_0(\mathsf{D}^{\mathsf{b}}(\mathcal{E}))$.*

Combining Theorem 1.1, Theorem 2.7 and this lemma, we have the following corollary.

Corollary 3.3. *Let $\mathcal{E}$ be an essentially small exact category with either a generator or a cogenerator $\mathcal{G}$. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{E}$*$\}$,
(2) $\{$*dense triangulated subcategories of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ containing $\mathcal{G}$*$\}$, *and*
(3) $\{$*subgroups of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$*$\}$.

Taking $\mathcal{G} = \mathsf{proj}\,\mathcal{E}$ in this corollary gives the dense version of the following theorem due to Krause and Stevenson:

Theorem 3.4. [15, Theorem 1] *Let $\mathcal{E}$ be an exact category with enough projective objects. Then there is one-to-one correspondence between*

(1) $\{$*thick subcategories of $\mathcal{E}$ containing $\mathsf{proj}\,\mathcal{E}$*$\}$ *and*
(2) $\{$*thick triangulated subcategories of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ containing $\mathsf{proj}\,\mathcal{E}$*$\}$.

Next, we give a more concrete corollary.

Let S be an *Iwanaga-Gorenstein ring* (i.e. S is noetherian on both sides and S is of finite injective dimension as a left S-module and a right S-module). Let us give several remarks about Iwanaga-Gorenstein rings (cf. [6, 25]).

Remark 3.5. (1) [6, Lemma 4.4.2] We say that a finitely generated left S-module X is *maximal Cohen-Macaulay* if $\mathsf{Ext}^i_S(X, S) = 0$ for all integers $i > 0$. $\mathsf{CM}(S)$ denotes the subcategory of $\mathsf{mod}\, S$ consisting of maximal Cohen-Macaulay S-modules. Then it is a Frobenius category, and hence, its stable category $\underline{\mathsf{CM}}(S)$ is triangulated.

(2) Natural inclusions $\mathsf{CM}(S) \hookrightarrow \mathsf{mod}\, S \hookrightarrow \mathsf{D^b}(\mathsf{mod}\, S)$ induce isomorphisms

$$K_0(\mathsf{CM}(S)) \cong K_0(\mathsf{mod}\, S) \cong K_0(\mathsf{D^b}(\mathsf{mod}\, S)).$$

Here, the first isomorphism is shown in [25, Lemma 13.2] and the second isomorphism is by Lemma 3.2

(3) [6, Theorem 4.4.1] Composition of the natural inclusion $\mathsf{CM}(S) \hookrightarrow \mathsf{D^b}(\mathsf{mod}\, S)$ and the quotient functor $\mathsf{D^b}(\mathsf{mod}\, S) \to \mathsf{D_{sg}}(S) := \mathsf{D^b}(\mathsf{mod}\, S)/\mathsf{K^b}(\mathsf{proj}(\mathsf{mod}\, S))$ induces a triangle equivalence

$$\underline{\mathsf{CM}}(S) \cong \mathsf{D_{sg}}(S).$$

Corollary 3.6. *Let S be an Iwanaga-Gorenstein ring. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense resolving subcategories of* $\mathsf{CM}(S)\}$,
(2) $\{$*dense resolving subcategories of* $\mathsf{mod}\, S\}$,
(3) $\{$*dense triangulated subcategories of* $\mathsf{D^b}(\mathsf{mod}\, S)$ *containing* $\mathsf{proj}(\mathsf{mod}\, S)\}$,
(4) $\{$*dense triangulated subcategories of* $\underline{\mathsf{CM}}(S) \cong \mathsf{D_{sg}}(S)\}$,
(5) $\{$*subgroups of* $K_0(\mathsf{mod}\, S)$ *containing the image of* $\mathsf{proj}(\mathsf{mod}\, S)\}$, *and*
(6) $\{$*subgroups of* $K_0(\underline{\mathsf{CM}}(S))\}$.

Proof. One-to-one correspondences among (1), (2), (3), and (5) follow from the above remarks and Corollary 3.3. The bijection between (4) and (6) follows from Thomason's result, Theorem 1.1. Thus, we show the one-to-one correspondence between (5) and (6).

The localization sequence

$$\mathsf{D^b}(\mathsf{proj}(\mathsf{mod}\, S)) \to \mathsf{D^b}(\mathsf{mod}\, S) \to \mathsf{D_{sg}}(S)$$

yields the exact sequence

$$K_0(\mathsf{D^b}(\mathsf{proj}(\mathsf{mod}\, S))) \to K_0(\mathsf{D^b}(\mathsf{mod}\, S)) \to K_0(\mathsf{D_{sg}}(S)) \to 0.$$

The equivalence $\mathsf{D_{sg}}(S) \cong \underline{\mathsf{CM}}(S)$ and Lemma 3.2 turns this sequence into the exact sequence

$$K_0(\mathsf{proj}(\mathsf{mod}\, S)) \to K_0(\mathsf{mod}\, S) \to K_0(\underline{\mathsf{CM}}(S)) \to 0.$$

Then, the equivalence is clear. ■

In the last two corollaries, we constructed a triangulated category $\mathsf{D^b}(\mathcal{E})$ from an given exact category $\mathcal{E}$ and discussed their dense subcategories. Next, we consider the opposite direction. More precisely, we construct an abelian category from a given triangulated category, and then we discuss their dense subcategories.

Let us recall the definition and some basic properties of t-structures; for details, see [11].

Definition 3.7. (1) A *t-structure* on $\mathcal{T}$ is a pair $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ of subcategories in $\mathcal{T}$ satisfying the following conditions:
(i) $\mathsf{Hom}_{\mathcal{T}}(\mathcal{T}^{\le -1}, \mathcal{T}^{\ge 0}) = 0$.
(ii) For any object $X \in \mathcal{T}$, there exists an exact triangle $X' \to X \to X'' \to X'[1]$ in $\mathcal{T}$ with $X' \in \mathcal{T}^{\le -1}$ and $X'' \in \mathcal{T}^{\ge 0}$.
(iii) $\mathcal{T}^{\le -1} \subset \mathcal{T}^{\le 0}$ and $\mathcal{T}^{\ge 0} \subset \mathcal{T}^{\ge -1}$.
Here, $\mathcal{T}^{\le -n} := \mathcal{T}^{\le 0}[n]$ and $\mathcal{T}^{\ge -n} := \mathcal{T}^{\ge 0}[n]$. Moreover, the intersection $\mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$ has the structure of an abelian category and we call it the *heart* of the t-structure.
(2) A t-structure $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ on $\mathcal{T}$ is called *bounded* if $\mathcal{T} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{T}^{\le i} \cap \mathcal{T}^{\ge j}$.

Example 3.8. Let $\mathcal{A}$ be an abelian category and put

$$\mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\le 0} := \{X \in \mathsf{D}^{\mathsf{b}}(\mathcal{A}) \mid \mathsf{H}^i(X) = 0 \ (\forall i > 0)\},$$
$$\mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\ge 0} := \{X \in \mathsf{D}^{\mathsf{b}}(\mathcal{A}) \mid \mathsf{H}^i(X) = 0 \ (\forall i < 0)\}.$$

Then $(\mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\le 0}, \mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\ge 0})$ defines a bounded t-structure on $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ and its heart is $\mathcal{A}$.

The next proposition is a variant of Lemma 3.2.

Proposition 3.9. [19] *Let $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ be a bounded t-structure on $\mathcal{T}$ with heart $\mathcal{A}$. Then the inclusion functor induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{T})$.*

From this proposition and Theorem 2.7, we have the following corollary.

Corollary 3.10. *Let $\mathcal{T}$ be an essentially small triangulated category, $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ a bounded t-structure on $\mathcal{T}$ with heart $\mathcal{A}$, and either a generator or a cogenerator $\mathcal{G}$ of $\mathcal{A}$. Then there are one-to-one correspondences among the following sets:*
(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{A}$*$\}$,
(2) $\{$*dense triangulated subcategories of $\mathcal{T}$ containing $\mathcal{G}$*$\}$, *and*
(3) $\{$*subgroups of $K_0(\mathcal{T})$ containing the image of $\mathcal{G}$*$\}$.

4. Examples

In this section, we give some examples of module categories which have only finitely many dense resolving subcategories.

Let us start with the following remark.

Remark 4.1. Let L be an abelian group. Then there are only finitely many subgroups of L if and only if L is a finite group. Indeed, ‘if part’ is clear. Suppose that there are only finitely many subgroups of L. Then L is an noetherian $\mathbb{Z}$-module and in particular, finitely generated. Therefore, there is an isomorphism

$$L \cong \mathbb{Z}^{\oplus r} \oplus \bigoplus_{i=1}^{n} (\mathbb{Z}/n\mathbb{Z})^{\oplus m_i},$$

where r, n and m_i are non-negative integers. We obtain $r = 0$ due to our assumption as $\mathbb{Z}$ has infinitely many subgroups. For this reason, L is isomorphic to a finite direct sum of finite abelian groups, and thus is a finite group.

From this remark and Theorem 2.7, for a left noetherian ring A, the following two conditions are equivalent:

(1) There are only finitely many dense resolving subcategories of $\mathsf{mod}\, A$.
(2) $K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle$ is a finite group.

4.1. **The case of finite dimensional algebras.** First we consider the case of finite dimensional algebras. Let A be a basic finite dimensional algebra over a field k with a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents. Denote $S_i := Ae_i/\mathsf{rad}_A(Ae_i)$ by the simple A-module corresponds to e_i. Then by [4, Theorem 3.5], $\{[S_1], \dots, [S_n]\}$ forms a free basis of the Grothendieck group $K_0(\mathsf{mod}\, A)$, and hence there is an isomorphism of abelian groups:

$$K_0(\mathsf{mod}\, A) \cong \mathbb{Z}^{\oplus n}.$$

The Cartan matrix of A is an $n \times n$-matrix $C_A := (\mathsf{dim}_k\, e_iAe_j)_{i,j=1,\dots,n}$. Then the above isomorphism induces the following isomorphism (see [4, Proposition 3.8]).

$$K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle \cong \mathsf{Coker}(\mathbb{Z}^{\oplus n} \xrightarrow{C_A} \mathbb{Z}^{\oplus n}).$$

Therefore if C_A has elementary divisors $(m_1, \cdots, m_r, 0, \cdots, 0)$, then we obtain a decomposition:

$$K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle \cong \mathbb{Z}^{\oplus n-r} \oplus \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_r),$$

where $m_1, \dots, m_r$ are not zero. Furthermore, one has

$$\mathsf{det}\, C_A = \begin{cases} 0 & (r < n) \\ m_1 \cdot m_2 \cdots m_n & (r = n). \end{cases}$$

As a result, the abelian group $K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle$ is a finite group if and only if the determinant of C_A is not zero.

From this argument, we have the following corollary.

Corollary 4.2. *Let A be a basic finite dimensional algebra over a field k. Then $\mathsf{mod}\, A$ has only finitely many dense resolving subcategories if and only if its Cartan matrix has non-zero determinant. This is the case, the number of dense resolving subcategories is $d(m_1) \cdots d(m_n)$. Here, $(m_1, \dots, m_n)$ are elementary divisors of C_A and $d(l)$ denotes the number of divisors of l.*

Remark 4.3. For the case of gentle algebras, Holm [12] gives a characterization of algebras with non-zero Cartan determinant $\mathsf{det}\, C_A$.

4.2. **The case of simple singularities.** Next we consider the case of simple singularities. Let k be an algebraically closed field of characteristic 0. We say that a commutative noetherian local ring $R := k[[x, y, z]]/(f)$ has a *simple (surface) singularity* if f is one of the following form:

$$\begin{array}{ll} (\mathsf{A}_n) & x^2 + y^{n+1} + z^2 \ (n \geq 1), \\ (\mathsf{D}_n) & x^2 y + y^{n-1} + z^2 \ (n \geq 4), \\ (\mathsf{E}_6) & x^3 + y^4 + z^2, \\ (\mathsf{E}_7) & x^3 + xy^3 + z^2, \\ (\mathsf{E}_8) & x^3 + y^5 + z^2. \end{array}$$

In this case, the Grothendieck group of $\mathsf{mod}\, R$ is given as follows (see [25, Proposition 13.10]):

	$K_0(\mathsf{mod}\, R)$	$\#\{$ dense resolv. subcat. of $\mathsf{mod}\, R\}$
(A_n)	$\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$	the number of divisors of $n+1$
(D_n) $(n = \text{even})$	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	5
(D_n) $(n = \text{odd})$	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	3
(E_6)	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	2
(E_7)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2
(E_8)	$\mathbb{Z}$	1

Here, $\mathbb{Z}$ appearing in $K_0(\mathsf{mod}\, R)$ is generated by $[R]$. Owing to Theorem 2.7, there are only finitely many dense resolving subcategories of $\mathsf{mod}\, R$. Hence the following natural question arises.

Question 4.4. Let R be a Gorenstein local ring of dimension two. Then does the condition $\#\{$dense resolving subcategories of $\mathsf{mod}\, R\} < \infty$ imply that R has a simple singularity?

Remark 4.5. 1-dimensional simple singularities may have infinitely many dense resolving subcategories (see [25, Proposition 13.10]).

Let R be a noetherian normal local domain with residue field k. Denote by $\mathsf{Cl}(R)$ the divisor class group of R. Then there is a surjective homomorphism

$$u = \begin{pmatrix} \mathsf{rk} \\ c_1 \end{pmatrix} : K_0(\mathsf{mod}\, R) \to \mathbb{Z} \oplus \mathsf{Cl}(R),$$

where rk is the *rank function* and c_1 is the *first Chern class*. Moreover, $u([R]) = {}^t(1, 0)$ and the kernel of u is the subgroup of $K_0(\mathsf{mod}\, R)$ generated by modules of codimension at least 2; see [5]. In particular, if R is a 2-dimensional noetherian normal local domain with residue field k, we obtain a short exact sequence

$$0 \to \langle [k] \rangle \to K_0(\mathsf{mod}\, R) \xrightarrow{\binom{\mathsf{rk}}{c_1}} \mathbb{Z} \oplus \mathsf{Cl}(R) \to 0$$

of abelian groups. This sequence induces the following short exact sequence since $\mathsf{rk}(R) = 1$ and $c_1(R) = 0$:

$$0 \to \langle [k], [R] \rangle / \langle [R] \rangle \to K_0(\mathsf{mod}\, R)/\langle [R] \rangle \xrightarrow{c_1} \mathsf{Cl}(R) \to 0.$$

Therefore, we have an isomorphism

$$\mathsf{Cl}(R) \cong K_0(\mathsf{mod}\, R)/\langle [k], [R] \rangle$$

and the following result is deduced from Theorem 2.7:

Theorem 4.6. *Let R be a noetherian normal local domain of dimension two. Then there is a one-to-one correspondence*

$$\left\{ \begin{matrix} \textit{dense resolving subcategories of } \mathsf{mod}\, R \\ \textit{containing } k \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \{ \textit{subgroups of } \mathsf{Cl}(R) \}$$

where f and g are given by $f(\mathcal{X}) := \langle c_1(X) \mid X \in \mathcal{X} \rangle$ and $g(H) := \{ X \in \mathsf{mod}\, R \mid c_1(X) \in H \}$ respectively.

The following answers Question 4.4 for domains.

Corollary 4.7. *Let R be a 2-dimensional complete non-regular Gorenstein normal local domain with algebraically closed residue field k of characteristic 0. Then the following are equivalent:*

(1) *R has a simple singularity.*
(2) *There are only finitely many dense resolving subcategories of $\mathsf{mod}\, R$.*
(3) *There are only finitely many dense resolving subcategories of $\mathsf{mod}\, R$ containing k.*

Proof. (1) $\Rightarrow$ (2): If R has a simple singularity, then $K_0(\mathsf{mod}\, R)/\langle [R] \rangle$ is a finite group; see [25, Proposition 13.10]. Thus, Theorem 2.7 shows that $\mathsf{mod}\, R$ has only finitely many dense resolving subcategories.

(2) $\Rightarrow$ (3): This implication is trivial.

(3) $\Rightarrow$ (1): From Theorem 4.6, $\mathsf{Cl}(R)$ has finitely many subgroups. Therefore, $\mathsf{Cl}(R)$ is a finite group, and thus by [9, Corollary 3.3] we have $\Omega\mathsf{CM}(R) = \mathsf{add}\, G$ for some module G, where $\Omega\mathsf{CM}(R)$ stands for the category of first syzygies of maximal Cohen-Macaulay R-modules. Now, since R is Gorenstein, $\mathsf{CM}(R) = \Omega\mathsf{CM}(R)$ has only finitely many indecomposable objects up to isomorphism. Consequently, R has a simple singularity from [25, Theorem 8.10]. ■

Example 4.8. Let R be a 2-dimensional simple singularity of type (A_1). Namely, $R = k[[x, y, z]]/(x^2 + y^2 + z^2)$. Then the indecomposable maximal Cohen-Macaulay R-modules are R and the ideal $I = (x + \sqrt{-1}y, z)$ up to isomorphism. Thus, every maximal Cohen-Macaulay module is of the form $R^{\oplus n} \oplus I^{\oplus m}$. Then the dense resolving subcategories of $\mathsf{mod}\, R$ are:

- $\mathsf{mod}\, R$, and
- $\{ M \in \mathsf{mod}\, R \mid \Omega^2 M \cong R^{\oplus n} \oplus I^{\oplus 2m} \text{ for some } m, n \in \mathbb{Z}_{\geq 0} \}$.

Proof. Set $G := K_0(\mathsf{mod}\, R)$ and let H be the subgroup generated by $[R]$.

First note that there is a non-split short exact sequence $0 \to I \to R^{\oplus 2} \to I \to 0$, see [25, Chapter 10]. Therefore, $[R]$ and $[I]$ satisfy $2[R] = 2[I]$ in G. Moreover, the isomorphism $G \cong K_0(\mathsf{CM}(R))$ shows that G and H are only subgroups of G containing $[R]$.

Using the notation of Theorem 2.7, we know that $g(G) = \mathsf{mod}\, R$. It thus suffices to show that $g(H) = \mathcal{X}$. Let M be an object of $\mathcal{X}$. From the exact sequence $0 \to \Omega^2 M \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0$, one has

$$[M] \equiv [\Omega^2 M] \equiv 0 \text{ mod } H.$$

This shows that $M \in g(H)$. Next, take $M \notin \mathcal{X}$. Then $\Omega^2 M \cong R^{\oplus n} \oplus I^{\oplus (2m+1)}$ for some $n, m \in \mathbb{Z}_{\geq 0}$. Using the similar argument, one has

$$[M] \equiv [\Omega^2 M] \equiv (2m+1)[I] \equiv [I] \text{ mod } H.$$

Hence if $[M]$ is in H, then so is $[I]$. This gives a contradiction to $G \neq H$. Therefore, $[M]$ cannot be in H. Thus, we are done. ∎

Acknowledgments. The author is grateful to his supervisor Ryo Takahashi for his many helpful comments. Also, the author very much thank the anonymous referee for his/her careful reading, valuable comments and helpful suggestions.

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