THE STRUCTURE OF PREENVELOPES WITH RESPECT TO MAXIMAL COHEN-MACAULAY MODULES

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Abstract. This paper studies the structure of special preenvelopes and envelopes with respect to maximal Cohen-Macaulay modules. We investigate the structure of them in terms of their kernels and cokernels. Moreover, using this result, we also study the structure of special proper coresolutions with respect to maximal Cohen-Macaulay modules over a Henselian Cohen-Macaulay local ring.

1. Introduction

Throughout this paper, we assume that $(R, \mathfrak{m}, k)$ is a d-dimensional Cohen-Macaulay local ring with canonical module ω. All R-modules are assumed to be finitely generated. Denote by $\mathsf{mod}R$ the category of finitely generated R-modules and by MCM the full subcategory of $\mathsf{mod}R$ consisting of maximal Cohen-Macaulay R-modules.

We define $(-)^\dagger := \mathsf{Hom}_R(-, \omega)$ and $\delta_M : M \to M^{\dagger\dagger}$ as a natural homomorphism for an R-module M. Note that if M is maximal Cohen-Macaulay, δ_M is an isomorphism.

Let $\mathcal{X}$ be a full subcategory of $\mathsf{mod}R$. The notion of $\mathcal{X}$-(pre)covers and $\mathcal{X}$-(pre)envelopes have been playing an important role in the representation theory of algebras; see [4–6, 8, 9, 13] for instance. For $\mathcal{X} = \mathsf{MCM}$, a celebrated theorem due to Auslander and Buchweitz [3] says that for any R-module M, there exists a short exact sequence

$$0 \to Y \xrightarrow{f} X \xrightarrow{\pi} M \to 0$$

where X is maximal Cohen-Macaulay and Y has finite injective dimension. The map π is called a maximal Cohen-Macaulay approximation of M. Then π is an MCM-precover of M, and is an MCM-cover if Y and X have no non-zero common direct summand via f. If R is Henselian, every R-module has an MCM-cover; see [12, 15].

In this paper we mainly study the MCM-envelope, and the MCM-preenvelope which is called special. A result of Holm [10, Theorem A] states that every R-module has a special MCM-preenvelope, and if R is Henselian, every R-module has an MCM-envelope. It is natural to ask when a given homomorphism is a special MCM-preenvelope or an MCM-envelope, and we give an answer to this question by using the kernels and cokernels. Our first main result is the following theorem.

Theorem 1.1. *Let $\mu : M \to X$ be an R-homomorphism such that X is maximal Cohen-Macaulay.*

(1) *The following are equivalent.*
  (a) *μ is a special MCM-preenvelope of M.*
  (b) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$ *and* $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$.

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(c) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, *and there exists an exact sequence* $0 \to S \to \mathsf{Coker}\,\mu \to T \to U \to 0$ *such that*
- $\mathsf{codim}\,S > 1$,
- $\mathsf{codim}\,U > 2$,
- T *satisfies* (S_2),
- $T^\dagger$ *has finite injective dimension and satisfies* (S_3).

(2) *The following are equivalent if* R *is Henselian.*

(a) μ *is an* MCM*-envelope of* M.

(b) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$ *and* $\mathsf{Coker}(\mu^{\dagger\dagger})$ *has no free summand.*

(c) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, *and there exists an exact sequence* $0 \to S \to \mathsf{Coker}\,\mu \xrightarrow{u} T \to U \to 0$ *such that*
- $\mathsf{codim}\,S > 1$,
- $\mathsf{codim}\,U > 2$,
- T *satisfies* (S_2),
- $T^\dagger$ *has finite injective dimension and satisfies* (S_3),
- $\mathsf{Im}\,u$ *has no non-zero free summand.*

The conditions (c) in (1) and (2) not only clarify the structure of special MCM-preenvelopes and MCM-envelopes, but also have the advantage that they do not contain vanishing conditions of Ext modules, which are in general hard to verify. We construct concrete examples of MCM-(pre)envelopes by using Theorem 1.1. Moreover, applying this result, we give another characterization of special MCM-preenvelopes in terms of the existence of certain complexes, which is our second main result.

Theorem 1.2. *Let* $\mu : M \to X$ *be an* R*-homomorphism such that* X *is maximal Cohen-Macaulay. Then the following are equivalent.*

(1) μ *is a special* MCM*-preenvelope of* M.

(2) *There exists an* R*-complex* $C = (0 \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{d-3}} C^{d-2} \to 0)$ *with* C^i *free for* $1 \le i \le d-2$ *and* $d^{-1} = \mu$ *such that* $\mathsf{codim}\,\mathsf{H}^i(C) > i+1$ *for any* i.

A special proper MCM-coresolution (resp. minimal proper MCM-coresolution) is such a complex that is built by taking special MCM-preenvelopes (resp. MCM-envelopes) repeatedly. Using Theorem 1.1, we prove the following result on the structure of special proper MCM-coresolutions as our third main result.

Theorem 1.3. *Suppose that* R *is Henselian. Let* M *be an* R*-module and*

$$0 \to M \xrightarrow{d^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots$$

a special proper MCM*-coresolution of* M. *Put* $\mu^0 := d^0$ *and let* $\mu^i : \mathsf{Coker}\,d^{i-1} \to X^i$ *be the induced homomorphisms. Then for each* i *one has* $\mathsf{codim}(\mathsf{Ker}\,\mu^i) > i$, *and there exists an exact sequence*

$$0 \to S^i \to \mathsf{Coker}\,\mu^i \to T^i \to U^i \to 0$$

such that
- $\mathsf{codim}\,S^i > i+1$,
- $\mathsf{codim}\,U^i > i+2$,
- T^i *satisfies* (S_2),
- $(T^i)^\dagger$ *has finite injective dimension and satisfies* (S_{i+3}).

We should remark that our Theorem 1.1 guarantees that the converse of the statement of Theorem 1.3 also holds: If a complex of R-modules

$$0 \to M \xrightarrow{d^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots$$

with each X^i maximal Cohen-Macaulay satisfies the conditions in the conclusion of Theorem 1.3, then this is a special proper MCM-coresolution of M. Furthermore, it turns out that Theorem 1.3 recovers a main theorem of Holm [10, Theorem C] in the Henselian case.

2. Preliminaries

In this section, we give several basic definitions and remarks for later use.

Definition 2.1. Let M be an R-module. Then, we set $\mathsf{codim}\, M := d - \mathsf{dim}\, M$ and call this the *codimension* of M.

We should remark that

$$\mathsf{codim}\, M = \mathsf{min}\{\mathsf{ht}\,\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0\}$$

from the definition of $\mathsf{dim}\, M$. Therefore, if $M_{\mathfrak{p}} = 0$ for any prime ideals $\mathfrak{p}$ with $\mathsf{ht}\,\mathfrak{p} \leq n$, we have $\mathsf{codim}\, M > n$.

Definition 2.2. Let $\mathcal{X}$ be a full subcategory of $\mathsf{mod} R$ and M an R-module.
(1) Let $\pi : X \to M$ be an R-homomorphism such that $X \in \mathcal{X}$.
 (a) π is called an *$\mathcal{X}$-precover* of M if

$$\mathsf{Hom}_R(X', \pi) : \mathsf{Hom}_R(X', X) \to \mathsf{Hom}_R(X', M)$$

 is an epimorphism for any $X' \in \mathcal{X}$.
 (b) π is called a *special $\mathcal{X}$-precover* of M if π is an $\mathcal{X}$-precover and satisfies $\mathsf{Ext}^1_R(\mathcal{X}, \mathsf{Ker}\,\pi) = 0$.
 (c) π is called an *$\mathcal{X}$-cover* of M if π is an $\mathcal{X}$-precover and for any $\phi \in \mathsf{End}_R(X)$, $\pi\phi = \pi$ implies ϕ is an automorphism.
(2) Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathcal{X}$.
 (a) μ is called an *$\mathcal{X}$-preenvelope* of M if

$$\mathsf{Hom}_R(\mu, X') : \mathsf{Hom}_R(X, X') \to \mathsf{Hom}_R(M, X')$$

 is an epimorphism for any $X' \in \mathcal{X}$.
 (b) μ is called a *special $\mathcal{X}$-preenvelope* of M if μ is an $\mathcal{X}$-preenvelope and satisfies $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathcal{X}) = 0$.
 (c) μ is called an *$\mathcal{X}$-envelope* of M if μ is an $\mathcal{X}$-preenvelope and for any $\phi \in \mathsf{End}_R(X)$, $\phi\mu = \mu$ implies ϕ is an automorphism.

Remark 2.3. (1) $\mathcal{X}$-precovers are not necessarily epimorphisms in general. If $\mathcal{X}$ contains R, then every $\mathcal{X}$-precover is an epimorphism.
(2) Let M be an R-module. An $\mathcal{X}$-cover of M is unique in the following sense: for two $\mathcal{X}$-covers $\pi : X \to M$ and $\pi' : X' \to M$ of M, there exists an isomorphism $\phi : X \to X'$ such that $\pi'\phi = \pi$. Dually, an $\mathcal{X}$-envelope of M is unique.
(3) By definition, a special $\mathcal{X}$-precover is an $\mathcal{X}$-precover. If $\mathcal{X}$ is closed under extensions, then an $\mathcal{X}$-cover is a special $\mathcal{X}$-precover: this result is called Wakamatsu's lemma (see [14]). The statement where "cover" is replaced with "envelope" also holds true.

(4) Let M be an R-module. If $\mathsf{Ext}_R^1(\mathsf{MCM}, M) = 0$, then M has finite injective dimension since $\mathsf{Ext}_R^{d+1}(k, M) \cong \mathsf{Ext}_R^1(\Omega^d k, M) = 0$. Here, Ω denotes the syzygy functor. On the other hand, if M has finite injective dimension, then $\mathsf{Ext}_R^1(\mathsf{MCM}, M) = 0$; see [3]. Therefore, a special MCM-precover of M is nothing but a maximal Cohen-Macaulay approximation of M.

Definition 2.4. Let M be an R-module, and

$$(*) \qquad 0 \to M \xrightarrow{d^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots$$

be an R-complex with $X^i \in \mathsf{MCM}$ for each i. Put $\mu^0 := d^0$, and let $\mu^i : \mathsf{Coker}\, d^{i-1} \to X^i$ be the induced morphisms for $i \geq 1$. If the μ^i are MCM-preenvelopes (resp. special MCM-preenvelopes; resp. MCM-envelopes), we call (*) a *proper* MCM-*coresolution* (resp. a *special proper* MCM-*coresolution*; resp. a *minimal proper* MCM-*coresolution*). By virtue of [10, Theorem A], we can construct a special proper MCM-coresolution of M, and if R is Henselian, we can construct a minimal proper MCM-coresolution of M for any $M \in \mathsf{mod}R$.

Definition 2.5. Let M and N be R-modules. We define $\mathsf{rad}_R(M, N)$ as the subgroup of $\mathsf{Hom}_R(M, N)$ consisting of homomorphisms f which satisfy the following condition: there is no non-zero direct summand M' of M such that M' is isomorphic to a direct summand of N via f. This definition is equivalent to the definition of rad_R in [11] if R is Henselian.

Remark 2.6. Let $\mathcal{X}$ be a full subcategory of $\mathsf{mod}R$, and let $0 \to K \xrightarrow{f} X \xrightarrow{\pi} M$ be an exact sequence in $\mathsf{mod}R$, where π is an $\mathcal{X}$-precover. Suppose that R is Henselian. Then π is an $\mathcal{X}$-cover if and only if $f \in \mathsf{rad}_R(K, X)$. For the proof, see [12, Proposition 2.4].

3. The structure of special MCM-preenvelopes

In this section, we prove our main result on the structure of MCM-(pre)envelopes. We give several lemmas used in the proof of the main theorem. The first one is due to Holm [10, Lemma 3.2, Proposition 3.3].

Lemma 3.1. (1) *Let M be an R-module. Then $\mathsf{Ext}_R^1(M, \mathsf{MCM}) = 0$ if and only if $\mathsf{Ext}_R^1(M, \omega) = 0 = \mathsf{Ext}_R^1(\mathsf{MCM}, M^\dagger)$.*

(2) *If $\mu : M \to X$ is an MCM-preenvelope (resp. a special MCM-preenvelope; resp. an MCM-envelope), then $\mu^\dagger : X^\dagger \to M^\dagger$ is an MCM-precover (resp. a special MCM-precover; resp. an MCM-cover).*

Lemma 3.2. *Let C be an R-module. Consider an exact sequence $0 \to L \xrightarrow{f} C \xrightarrow{g} N \to 0$ with $\mathsf{codim}\, L > 0$ such that N satisfies (S_1). Such an exact sequence is, if it exists, unique up to isomorphisms of complexes.*

Proof. Let $0 \to L' \xrightarrow{f'} C \xrightarrow{g'} N' \to 0$ be a short exact sequence satisfying the same condition. From [1, Proposition 3.1], N is 1-ω-torsion free, in other words, δ_N is a monomorphism. Therefore, N can be embedded in $\omega^{\oplus r}$ for some integer r. Note that $\mathsf{Ass}\,\mathsf{Hom}_R(L', \omega) = \mathsf{Supp}\, L' \cap \mathsf{Ass}\, \omega = \emptyset$: The first equality appears in [7, Exercise 1.2.27.] and the second follows from the assumption. Therefore, we have $\mathsf{Hom}_R(L', \omega) = 0$ and thus, $\mathsf{Hom}_R(L', N) = 0$. Therefore, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \xrightarrow{f'} & C & \xrightarrow{g'} & N' & \longrightarrow & 0 \\ & & {\scriptstyle s}\downarrow & & \| & & {\scriptstyle t}\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & C & \xrightarrow{g} & N & \longrightarrow & 0. \end{array}$$

Similarly, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & C & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & {\scriptstyle s'}\downarrow & & \| & & {\scriptstyle t'}\downarrow & & \\ 0 & \longrightarrow & L' & \xrightarrow{f'} & C & \xrightarrow{g'} & N' & \longrightarrow & 0. \end{array}$$

These two commutative diagrams yield $f's's = f'$ and since f' is a monomorphism, we conclude $s's = 1$. Using the same argument, one has $ss' = 1$. Thus, s is an isomorphism and so is t. ∎

Lemma 3.3. *Let* $0 \to U \to L \xrightarrow{\alpha} M \to V \to 0$ *be an exact sequence such that* $\mathsf{codim}\, U \geq i$ *and* $\mathsf{codim}\, V \geq i + 1$ *for an integer* i. *Then, we get an isomorphism* $\mathsf{Ext}^l_R(M, \omega) \cong \mathsf{Ext}^l_R(L, \omega)$ *for any integer* $l < i$.

Proof. From the short exact sequence $0 \to U \to L \to \mathsf{Im}\, \alpha \to 0$, we have an exact sequence

$$\mathsf{Ext}^{l-1}_R(U, \omega) \to \mathsf{Ext}^l_R(\mathsf{Im}\, \alpha, \omega) \to \mathsf{Ext}^l_R(L, \omega) \to \mathsf{Ext}^l(U, \omega).$$

Since $\mathsf{dim}\, U < d - l$, using the local duality theorem, we get $\mathsf{Ext}^l_R(U, \omega) \cong \mathsf{H}^{d-l}_{\mathfrak{m}}(U)^{\vee} = 0$ for $l < i$, where $(-)^{\vee}$ stands for the Matlis dual. Therefore, $\mathsf{Ext}^l_R(\mathsf{Im}\, \alpha, \omega) \cong \mathsf{Ext}^l_R(L, \omega)$ for $l < i$. Similarly, because we have $\mathsf{Ext}^l_R(V, \omega) = 0$ for $l < i + 1$ by the local duality theorem, using the same argument for $0 \to \mathsf{Im}\, \alpha \to M \to V \to 0$, we get an isomorphism $\mathsf{Ext}^l_R(M, \omega) \cong \mathsf{Ext}^l_R(\mathsf{Im}\, \alpha, \omega)$ for $l < i + 1$. Consequently, we obtain an isomorphism $\mathsf{Ext}^l_R(M, \omega) \cong \mathsf{Ext}^l_R(L, \omega)$ for $l < i$. ∎

Let M be an R-module and $P_1 \xrightarrow{f} P_0 \to M \to 0$ a projective presentation of M. Then we put $\mathsf{Tr}_{\omega} M := \mathsf{Coker}(f^{\dagger} : P_0^{\dagger} \to P_1^{\dagger})$ and call it an ω-transpose of M. The following lemma is well-known in the case where $\omega = R$. The proof of this lemma is shown along the same lines as in that of [2, Proposition 2.6].

Lemma 3.4. *Let* M *be an* R*-module. Then there exist isomorphisms*
(1) $\mathsf{Ker}\, \delta_M \cong \mathsf{Ext}^1_R(\mathsf{Tr}_{\omega} M, \omega)$ *and*
(2) $\mathsf{Coker}\, \delta_M \cong \mathsf{Ext}^2_R(\mathsf{Tr}_{\omega} M, \omega)$.

We are now ready to show our first main result.

Proof of Theorem 1.1. (a)⇒(b): By Lemma 3.1, $\mu^{\dagger} : X^{\dagger} \to M^{\dagger}$ is a special MCM-precover. In particular, $\mu^{\dagger}$ is an epimorphism. Since $\delta_X \mu = \mu^{\dagger\dagger} \delta_M$ and $\mu^{\dagger\dagger}$ is a monomorphism, $\mathsf{Ker}\, \delta_M \cong \mathsf{Ker}\, \mu$. Using Lemma 3.4, one has $\mathsf{Ker}\, \mu \cong \mathsf{Ext}^1_R(\mathsf{Tr}_{\omega} M, \omega)$. Because $\omega_{\mathfrak{p}}$ is an injective $R_{\mathfrak{p}}$-module for any minimal prime ideal $\mathfrak{p}$, we conclude $\mathsf{codim}(\mathsf{Ker}\, \mu) > 0$.

Next, consider the case (2). Then $\mu^{\dagger}$ is an MCM-cover, and hence there exists an exact sequence $0 \to Y \xrightarrow{f} X^{\dagger} \xrightarrow{\mu^{\dagger}} M^{\dagger} \to 0$ such that Y has finite injective dimension and

$f \in \mathsf{rad}_R(Y, X^\dagger)$. Applying $(-)^\dagger$, we get an exact sequence

$$0 \to M^{\dagger\dagger} \xrightarrow{\mu^{\dagger\dagger}} X^{\dagger\dagger} \xrightarrow{f^\dagger} Y^\dagger \to \mathsf{Ext}^1_R(M^\dagger, \omega) \to 0.$$

Then $f^\dagger$ can be decomposed into an epimorphism $g : X^{\dagger\dagger} \to \mathsf{Coker}(\mu^{\dagger\dagger})$ and a monomorphism $h : \mathsf{Coker}(\mu^{\dagger\dagger}) \to Y^\dagger$. Since $\mathsf{depth}_{R_\mathfrak{p}} (M^\dagger)_\mathfrak{p} \geq \mathsf{min}\{\mathsf{ht}\,\mathfrak{p}, 2\}$, $M^\dagger$ satisfies (S_2), and therefore we get $\mathsf{codim}(\mathsf{Ext}^1_R(M^\dagger, \omega)) > 2$. Hence $h^\dagger$ is an isomorphism by Lemma 3.3. By the depth lemma, Y satisfies (S_3) and hence [1, Proposition 3.1] implies that δ_Y is an isomorphism. Therefore, $f \in \mathsf{rad}_R(Y, X^\dagger)$ yields $f^{\dagger\dagger} = g^\dagger h^\dagger \in \mathsf{rad}_R(Y^{\dagger\dagger}, X^{\dagger\dagger\dagger})$, and hence $g^\dagger \in \mathsf{rad}_R((\mathsf{Coker}(\mu^{\dagger\dagger}))^\dagger, X^{\dagger\dagger\dagger})$. Because $g^\dagger \in \mathsf{rad}_R((\mathsf{Coker}(\mu^{\dagger\dagger}))^\dagger, X^{\dagger\dagger\dagger})$, $\mathsf{Coker}(\mu^{\dagger\dagger})$ and $X^{\dagger\dagger\dagger}$ has no non-zero common direct summand via $g^\dagger$. This shows that $X^{\dagger\dagger}$ and $\mathsf{Coker}(\mu^{\dagger\dagger})$ has no non-zero common direct summand via g. Consequently, $\mathsf{Coker}(\mu^{\dagger\dagger})$ has no non-zero free summand.

(b)⇒(c): By the local duality theorem, we have $(\mathsf{Ker}\,\mu)^\dagger = 0$. Since $(\mathsf{Ker}\,\mu)^\dagger = 0 = \mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \omega)$, $\mu^\dagger$ is an epimorphism. Taking a short exact sequence $0 \to Y \to X^\dagger \xrightarrow{\mu^\dagger} M^\dagger \to 0$ and applying $(-)^\dagger$ to this sequence, we obtain an exact sequence

$$0 \to M^{\dagger\dagger} \xrightarrow{\mu^{\dagger\dagger}} X^{\dagger\dagger} \to Y^\dagger \to \mathsf{Ext}^1_R(M^\dagger, \omega) \to 0.$$

Because $\delta_X \mu = \mu^{\dagger\dagger}\delta_M$, we get another exact sequence

$$\mathsf{Ker}(\mu^{\dagger\dagger}) = 0 \to \mathsf{Coker}\,\delta_M \to \mathsf{Coker}\,\mu \to \mathsf{Coker}(\mu^{\dagger\dagger}) \to 0.$$

Combining this two sequences, we have an exact sequence

$$0 \to \mathsf{Coker}\,\delta_M \to \mathsf{Coker}\,\mu \xrightarrow{\alpha} Y^\dagger \to \mathsf{Ext}^1_R(M^\dagger, \omega) \to 0.$$

We verify that this sequence satisfies the condition (c).

Since $\omega_\mathfrak{p}$ has injective dimension at most 1 for any prime ideal $\mathfrak{p}$ with $\mathsf{ht}\,\mathfrak{p} \leq 1$, $\mathsf{Coker}\,\delta_M \cong \mathsf{Ext}^2_R(\mathsf{Tr}_\omega, \omega)$ has codimension at least 2. As $M^\dagger$ satisfies (S_2), $\mathsf{codim}(\mathsf{Ext}^1_R(M^\dagger, \omega)) > 2$. Because Y satisfies (S_3), δ_Y is an isomorphism. Hence $(Y^\dagger)^\dagger \cong Y$ satisfies (S_3). By Lemma 3.1(1), one has $\mathsf{Ext}^1_R(\mathsf{MCM}, (\mathsf{Coker}\,\mu)^\dagger) = 0$, that is, $(Y^\dagger)^\dagger \cong (\mathsf{Coker}\,\mu)^\dagger$ has finite injective dimension. Consequently, μ satisfies the conditions in (c). Moreover, since $\mathsf{Im}\,\alpha \cong \mathsf{Coker}(\mu^{\dagger\dagger})$, the implication (b) ⇒ (c) also holds in the case of (2).

(c) ⇒ (a): First, we prove $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$. By assumption, there exists an exact sequence

$$0 \to S \to \mathsf{Coker}\,\mu \xrightarrow{\alpha} T \to U \to 0$$

which satisfies the conditions in (c). By the local duality theorem, we have $S^\dagger = 0 = \mathsf{Ext}^1_R(S, \omega)$, hence $\mathsf{Ext}^1_R(S, \mathsf{MCM}) = 0$. Therefore we have only to prove $\mathsf{Ext}^1_R(\mathsf{Im}\,\alpha, \mathsf{MCM}) = 0$. Since $(\mathsf{Im}\,\alpha)^\dagger \cong T^\dagger$ has finite injective dimension, one has $\mathsf{Ext}^1_R(\mathsf{MCM}, (\mathsf{Im}\,\mu)^\dagger) = 0$. The short exact sequence $0 \to \mathsf{Im}\,\alpha \to T \to U \to 0$ induces an exact sequence

$$\mathsf{Ext}^1_R(T, \omega) \to \mathsf{Ext}^1_R(\mathsf{Im}\,\alpha, \omega) \to \mathsf{Ext}^2_R(U, \omega).$$

Since $\mathsf{codim}\,U > 2$, $\mathsf{Ext}^2_R(U, \omega) = 0$. From [1, Proposition 3.1], we get $\mathsf{Ext}^1_R(T^{\dagger\dagger}, \omega) = 0$ and $T \cong T^{\dagger\dagger}$. Therefore, $\mathsf{Ext}^1_R(\mathsf{Im}\,\alpha, \omega) = 0$. Using Lemma 3.1, we conclude $\mathsf{Ext}^1_R(\mathsf{Im}\,\alpha, \mathsf{MCM}) = 0$.

Next, we show that μ is a special MCM-preenvelope. Let Z be a maximal Cohen-Macaulay R-module and $f : M \to Z$ an R-homomorphism. Since $\mathsf{Ass}\,\mathsf{Hom}_R(\mathsf{Ker}\,\mu, Z) =$

$\mathsf{Supp}(\mathsf{Ker}\,\mu) \cap \mathsf{Ass}\, Z = \emptyset$, f can be lifted to $\mathsf{Im}\,\mu$. Furthermore, because $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, Z) = 0$, the morphism $\mathsf{Im}\,\mu \to Z$ can be lifted to X. This shows that μ is an MCM-preenvelope.

Consider the case (2). Using [10, Theorem A], there exists an MCM-envelope $\mu' : M \to X'$. We show that μ is isomorphic to μ'. As we have already seen, μ is a special MCM-preenvelope. Hence there exists a commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\mu'} & X' & \longrightarrow & \mathsf{Coker}\,\mu' & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ M & \xrightarrow{\mu} & X & \longrightarrow & \mathsf{Coker}\,\mu & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \overline{g}} & & \\ M & \xrightarrow{\mu'} & X' & \longrightarrow & \mathsf{Coker}\,\mu' & \longrightarrow & 0 \end{array}$$

and since μ' is an MCM-envelope, gf and $\overline{g}\overline{f}$ are automorphisms. Consider the chain map:

$$\begin{array}{cccccccccc} F = (0 & \longrightarrow & M & \xrightarrow{\mu'} & X' & \longrightarrow & \mathsf{Coker}\,\mu' & \longrightarrow & 0\,) \\ {\scriptstyle u}\downarrow \quad \downarrow & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \downarrow \\ G = (0 & \longrightarrow & M & \xrightarrow{\mu} & X & \longrightarrow & \mathsf{Coker}\,\mu & \longrightarrow & 0\,). \end{array}$$

Then the cokernel of u is of the form $\mathsf{Coker}\,u = (0 \to 0 \to \mathsf{Coker}\,f \to \mathsf{Coker}\,\overline{f} \to 0)$ and by calculating the homologies of $\mathsf{Coker}\,u$, we conclude $\mathsf{Coker}\,f \cong \mathsf{Coker}\,\overline{f}$. Since f and f' are split monomorphisms, $\mathsf{Coker}\,f \cong \mathsf{Coker}\,\overline{f}$ is a maximal Cohen-Macaulay and $(\mathsf{Coker}\,f)^\dagger \cong (\mathsf{Coker}\,\overline{f})^\dagger$ has finite injective dimension by Lemma 3.1(1). This shows that $\mathsf{Coker}\,f \cong \mathsf{Coker}\,\overline{f}$ is a free R-module. From the assumption, there exists an exact sequence

$$0 \to S \to \mathsf{Coker}\,\mu \xrightarrow{\alpha} T \to U \to 0$$

which satisfies the conditions in (c). On the other hand, as we have shown in the proof of the implications (a)⇒(b)⇒(c), $\mathsf{Coker}\,\mu'$ also admits an exact sequence

$$0 \to S' \to \mathsf{Coker}\,\mu' \xrightarrow{\alpha'} T' \to U' \to 0$$

which satisfies the conditions in (c). Then we obtain two short exact sequences

$$0 \to S \to \mathsf{Coker}\,\mu \to \mathsf{Im}\,\alpha \to 0, \tag{1}$$

$$0 \to S' \to \mathsf{Coker}\,\mu' \to \mathsf{Im}\,\alpha' \to 0. \tag{2}$$

Since $\mathsf{Coker}\,\mu \cong \mathsf{Coker}\,\mu' \oplus \mathsf{Coker}\,\overline{f}$, we obtain a short exact sequence

$$0 \to S' \to \mathsf{Coker}\,\mu \to \mathsf{Im}\,\alpha' \oplus \mathsf{Coker}\,\overline{f} \to 0 \tag{2'}$$

from (2) by taking the direct sum with $\mathsf{Coker}\,\overline{f}$. By assumption, S and S' have codimension at least 2. Since T and T' satisfy (S_1), those submodules $\mathsf{Im}\,\alpha$ and $\mathsf{Im}\,\alpha'$ are also satisfy (S_1). Using Lemma 3.2, (1) and (2') are isomorphic as complexes. In particular, $\mathsf{Im}\,\alpha' \oplus \mathsf{Coker}\,\overline{f}$ is isomorphic to $\mathsf{Im}\,\alpha$. Because $\mathsf{Im}\,\alpha$ has no non-zero free summand, the free module $\mathsf{Coker}\,\overline{f}$ is 0. Consequently, μ is isomorphic to μ'. ■

Let us construct examples of special MCM-preenvelopes and MCM-envelopes by using Theorem 1.1.

Example 3.5. (1) Let R be a d-dimensional Cohen-Macaulay local ring with canonical module ω and $\underline{x} = x_1, x_2, \dots x_n$ an R-regular sequence with $n \geq 3$. Consider an exact sequence

$$0 \to M \xrightarrow{\mu} R^{\oplus n} \xrightarrow{(x_1,\dots,x_n)} R \to R/(\underline{x}) \to 0.$$

Since the cokernel of μ admits an exact sequence $0 \to \mathsf{Coker}\,\mu \to R \to R/(x) \to 0$ which satisfies the conditions in (c) in Theorem 1.1, $\mu : M \to R^{\oplus n}$ is a special MCM-preenvelope.

(2) Let R be a d-dimensional Cohen-Macaulay local domain with canonical module ω such that $d \geq 2$. Take an R-sequence x, y and a non-zero element $f \in R$. Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\binom{xf}{yf}} & R^{\oplus 2} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & R & \xrightarrow{\binom{x}{y}} & R^{\oplus 2} & \longrightarrow & (x,y) & \longrightarrow & 0. \end{array}$$

Using the snake lemma, we have a short exact sequence $0 \to R/(f) \to M \xrightarrow{g} (x, y) \to 0$. Let $\mu : M \to R$ be the composition of $g : M \to (x, y)$ and the inclusion $(x, y) \to R$. Then $\mathsf{Ker}\,\mu \cong \mathsf{Ker}\,g \cong R/(f)$ has codimension at least 1, and $\mathsf{Coker}\,\mu \cong R/(x, y)$ has codimension at least 2 and $R/(x, y)$ has no free summand. Therefore, μ is an MCM-envelope.

Remark 3.6. For each $Y \in \mathsf{mod}R$, Y has finite injective dimension if and only if Y admits an exact sequence $0 \to W_n \to W_{n-1} \to \cdots \to W_0 \to Y \to 0$ with $W_i \in \mathsf{add}\,\omega$ and $n = d - \mathsf{depth}_R Y$. Therefore, for an R-homomorphism $\pi : X \to M$ with $X \in \mathsf{MCM}$, the following are equivalent.

(1) π is a special MCM-precover.
(2) There exists an exact sequence $0 \to W_n \to W_{n-1} \to \cdots \to W_1 \to X \xrightarrow{\pi} M \to 0$ with $W_i \in \mathsf{add}\,\omega$ and $n = d - \mathsf{depth}_R M$.

Now, let us give a proof of Theorem 1.2

Proof of Theorem 1.2. (1) $\Rightarrow$ (2): Because of Lemma 3.1, $\mu^\dagger : X^\dagger \to M^\dagger$ is a special MCM-precover and there exists an exact sequence

$$(*) \qquad 0 \to W_{d-2} \xrightarrow{f_{d-2}} W_{d-3} \xrightarrow{f_{d-3}} \cdots \xrightarrow{f_2} W_1 \xrightarrow{f_1} X^\dagger \xrightarrow{\mu^\dagger} M^\dagger \to 0$$

such that $W_i \in \mathsf{add}\,\omega$. Applying $(-)^\dagger$, we obtain an R-complex

$$C = (0 \to M \xrightarrow{\mu = \delta_X^{-1}\mu^{\dagger\dagger}\delta_M} X \xrightarrow{f_1^\dagger \delta_X} W_1^\dagger \xrightarrow{f_2^\dagger} \cdots \xrightarrow{f_{d-2}^\dagger} W_{d-2}^\dagger \to 0).$$

Decompose (*) into short exact sequences $0 \to U_i \xrightarrow{u_i} W_i \xrightarrow{v_i} U_{i-1} \to 0$ $(0 \leq i \leq d-2)$ where $W_0 := X^\dagger$ and $U_{-1} := M^\dagger$. Applying $(-)^\dagger$ to these sequences, we obtain exact sequences $0 \to U_{i-1}^\dagger \xrightarrow{v_i^\dagger} W_i^\dagger \xrightarrow{u_i^\dagger} U_i^\dagger \to \mathsf{Ext}_R^1(U_{i-1}, \omega) \to 0$. Since $v_i^\dagger$ are monomorphisms,

$$\mathsf{H}^i(C) = \mathsf{Ker}\, f_{i+1}^\dagger / \mathsf{Im}\, f_i^\dagger = \mathsf{Ker}\, u_i^\dagger / \mathsf{Im}\, f_i^\dagger \cong \mathsf{Coker}\, u_{i-1}^\dagger \cong \mathsf{Ext}_R^1(U_{i-2}^\dagger, \omega)$$

for $1 \leq i \leq d-2$. From the exact sequence (*), $(U_{i-2})_{\mathfrak{p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module for any prime ideal $\mathfrak{p}$ with $\mathsf{ht}\,\mathfrak{p} \leq i+1$ by the depth lemma. Hence $\mathsf{codim}\,\mathsf{H}^i(C) > i+1$ for $i \geq 1$. For $i = -1, 0$,

$$\mathsf{H}^{-1}(C) = \mathsf{Ker}(\mu^{\dagger\dagger}\delta_M) = \mathsf{Ker}\,\delta_M \cong \mathsf{Ext}^1_R(\mathsf{Tr}_\omega M, \omega)$$
$$\mathsf{H}^0(C) = \mathsf{Ker}\,f_1^\dagger / \mathsf{Im}(\mu^{\dagger\dagger}\delta_M) \cong \mathsf{Coker}\,\delta_M \cong \mathsf{Ext}^2_R(\mathsf{Tr}_\omega M, \omega).$$

Therefore, $\mathsf{codim}\,\mathsf{H}^i(C) > i+1$ for any i.

(2) $\Rightarrow$ (1): First note that if $d \leq 2$, then this implication holds. Indeed, since $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$ and $\mathsf{codim}(\mathsf{Coker}\,\mu) > 1$, μ is a special MCM-preenvelope by Theorem 1.1.

Next, consider the case $d \geq 3$. Put $d^0 := \mu$ and denote by μ^i the induced homomorphism $\mathsf{Coker}\,d^{i-1} \to C^i$ for each $1 \leq i \leq d-2$. Note that there exists an exact sequence

$$0 \to \mathsf{H}^{i-1}(C) \to \mathsf{Coker}\,d^{i-1} \xrightarrow{\mu^i} C^i \to \mathsf{Coker}\,d^i \to 0.$$

Let us show $\mathsf{Ext}^1_R(\mathsf{Coker}\,d^i, \mathsf{MCM}) = 0$ for $0 \leq i \leq d-2$. By assumption, $\mathsf{Coker}\,d^{d-2} \cong \mathsf{H}^{d-2}(C)$ and $\mathsf{H}^{d-3}(C)$ have codimension at least 2. Hence $\mathsf{Ext}^1_R(\mathsf{Coker}\,d^{d-2}, \mathsf{MCM}) = 0 = \mathsf{Ext}^1_R(\mathsf{H}^{d-3}(C), \mathsf{MCM})$. From the exact sequence

$$0 \to \mathsf{Im}\,\mu^{d-2} \to C^{d-2} \to \mathsf{Coker}\,d^{d-2} \to 0,$$

we have $\mathsf{Ext}^1_R(\mathsf{Im}\,\mu^{d-2}, \mathsf{MCM}) = 0$. Hence one has $\mathsf{Ext}^1_R(\mathsf{Coker}\,d^{d-3}, \mathsf{MCM}) = 0$ from the exact sequence

$$0 \to \mathsf{H}^{d-3}(C) \to \mathsf{Coker}\,d^{d-3} \to \mathsf{Im}\,\mu^{d-2} \to 0.$$

Iterating this procedure, we get $\mathsf{Ext}^1_R(\mathsf{Coker}\,d^i, \mathsf{MCM}) = 0$ for $0 \leq i \leq d-2$. Since $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$ and $\mathsf{Ker}\,\mu \cong \mathsf{H}^{-1}(C)$ has positive codimension, μ is a special MCM-preenvelope. ■

4. The structure of special proper MCM-coresolutions

In this section, we study the structure of special proper MCM-coresolutions by using Theorem 1.1. From now on, we assume that R is Henselian. The following lemma is the key to prove our last theorem.

Lemma 4.1. *Let*

$$0 \to M \xrightarrow{d^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots$$

be a special proper MCM*-coresolution of an R-module M. Put $\mu^0 = d^0$, and let $\mu^i : \mathsf{Coker}\,d^{i-1} \to X^i$ be the induced morphisms for $i \geq 1$. Then the following holds:*

(1) $\mathsf{Coker}\,\mu^i$ *are unique up to free summands with respect to M and*

(2) $\mathsf{Ker}\,\mu^i$ *are unique up to isomorphisms with respect to M.*

Proof. Set $M^i := \mathsf{Coker}\,d^{i-1}$ $(i \geq 1)$ and $M^0 := M$.

(1): As we saw in the proof of the implication (c)$\Rightarrow$(a) in Theorem 1.1(2), for any R-module M, the cokernel of a special MCM-preenvelope of M is unique up to free summands. On the other hand, for a special MCM-preenvelope $\mu : M \to X$ and a free module F, $\mu \oplus F : M \oplus F \to X \oplus F$ is also a special MCM-preenvelope since special MCM-preenvelopes are characterized only by their kernels and cokernels by Theorem 1.1. Consequently, $\mathsf{Coker}\,\mu^i$ are unique up to free summands.

(2): Let M be an R-module and F a free module. Then the kernels of special MCM-preenvelopes of M and $M \oplus F$ are isomorphic by using Lemma 3.3. This shows (2). ■

Now, let us prove Theorem 1.3 given in the introduction.

Proof of Theorem 1.3. To prove this theorem, we have only to construct a such special proper MCM-coresolution.

Take a special MCM-preenvelope $\mu^0 : M \to X^0$. Then $\mathsf{Ker}\,\mu^0$ and $\mathsf{Coker}\,\mu^0$ satisfy the desired conditions by Theorem 1.1. For $i > 1$, assume that there exists an R-complex

$$0 \to M \xrightarrow{d^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{i-1}} X^{i-1}$$

which satisfies the desired conditions for $j < i$. By assumption, there exists an exact sequence

$$0 \to S^{i-1} \to \mathsf{Coker}\,\mu^{i-1} \xrightarrow{u} T^{i-1} \to U^{i-1} \to 0$$

which satisfies the conditions in the statement. Since $(T^{i-1})^\dagger$ has finite injective dimension, we can choose a short exact sequence $0 \to A \to W \to (T^{i-1})^\dagger \to 0$ where $W \in \mathsf{add}\,\omega$ and A has finite injective dimension. Applying $(-)^\dagger$, we have an exact sequence

$$0 \to T^{i-1} \xrightarrow{v} W^\dagger \to A^\dagger \to \mathsf{Ext}^1_R((T^{i-1})^\dagger, \omega) \to 0.$$

Set $\mu^i = vu : \mathsf{Coker}\,d^{i-1} \to W^\dagger$ and we show it satisfies the conditions in the statement. Then there exists an exact sequence

$$\mathsf{Ker}\,v = 0 \to \mathsf{Coker}\,u \to \mathsf{Coker}\,\mu^i \to \mathsf{Coker}\,v \to 0.$$

Splicing this sequence with $0 \to \mathsf{Coker}\,v \to A^\dagger \to \mathsf{Ext}^1_R((T^{i-1})^\dagger, \omega) \to 0$, we obtain an exact sequence

$$0 \to \mathsf{Coker}\,u \to \mathsf{Coker}\,\mu^i \to A^\dagger \to \mathsf{Ext}^1_R((T^{i-1})^\dagger, \omega) \to 0.$$

Set $S^i = \mathsf{Coker}\,u = U^{i-1}$, $T^i = A^\dagger$ and $U^i = \mathsf{Ext}^1_R((T^{i-1})^\dagger, \omega)$. By assumption, $\mathsf{Ker}\,\mu^i \cong \mathsf{Ker}\,u \cong S^{i-1}$ has codimension at least $i+1$ and $S^i = \mathsf{Coker}\,u = U^{i-1}$ has codimension at least $i+2$. Since $(T^{i-1})^\dagger$ satisfies (S_{i+2}), $((T^{i-1})^\dagger)_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module for any prime ideal $\mathfrak{p}$ with $\mathsf{ht}\,\mathfrak{p} \le i+2$. Hence $U^i = \mathsf{Ext}^1_R((T^{i-1})^\dagger, \omega)$ has codimension at least $i+3$. Note that $T^i = A^\dagger$ satisfies (S_2). Because A satisfies (S_{i+3}), δ_A is an isomorphism. Therefore, $(T^i)^\dagger = A^{\dagger\dagger} \cong A$ has finite injective dimension and satisfies (S_{i+3}). By induction on i, the proof of theorem is completed. ∎

The following result recovers the result [10, Theorem C] if R is Henselian. It is shown by examining the structure of an MCM-envelope concretely.

Corollary 4.2. *For any R-module M, the minimal proper* MCM*-coresolution of M has length at most $d-2$.*

Proof. For an R-module M, take a minimal proper MCM-coresolution

$$0 \to M \xrightarrow{\mu^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots .$$

Let $\mu^i : \mathsf{Coker}\,d^{i-1} \to X^i$ be the induced morphism. For $i = d-2$, there exists an exact sequence

$$0 \to S^{d-2} \to \mathsf{Coker}\,d^{d-2} \to T^{d-2} \to U^{d-2} \to 0$$

which satisfies the conditions in the statement of Theorem 1.1. Since U^{d-2} have codimension at least $d+1$, $U^{d-2} = 0$. On the other hand, $(T^{d-2})^\dagger$ satisfies (S_{d+1}) and has finite injective dimension, i.e., $(T^{d-2})^\dagger \in \mathsf{add}\,\omega$. As T^{d-2} satisfies (S_2), T^{d-2} is isomorphic to $(T^{d-2})^{\dagger\dagger}$, whence, free. Therefore, we have an exact sequence

$$0 \to S^{d-2} \to \mathsf{Coker}\,d^{d-2} \to T^{d-2} \to 0$$

where $\mathsf{codim}\, S^{d-2} \geq d$ and T^{d-2} is free. Since $\mu^{d-2} : \mathsf{Coker}\, d^{d-3} \to X^{d-2}$ is an MCM-envelope, $T^{d-2} = 0$. Therefore, we have $\mathsf{Coker}\, d^{d-2} \geq d$, in other words, $\mathsf{Coker}\, d^{d-2}$ has finite length. From [10, Proposition 4.1], $\mathsf{Coker}\, d^{d-2} \to 0$ is an MCM-envelope of $\mathsf{Coker}\, d^{d-2}$. This conclude that the minimal proper MCM-coresolution ends in X^{d-2}. ■

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