A new proof of the Bondal-Orlov reconstruction using Matsui spectra

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Abstract

In 2005, Balmer defined the ringed space $\operatorname{Spec}_{\otimes}\mathcal{T}$ for a given tensor triangulated category, while in 2023, the second author introduced the ringed space $\operatorname{Spec}_{\vartriangle}\mathcal{T}$ for a given triangulated category. In the algebro-geometric context, these spectra provided several reconstruction theorems using derived categories. In this paper, we prove that $\operatorname{Spec}_{\otimes^{\mathbb{L}}_X}\operatorname{Perf} X$ is an open ringed subspace of $\operatorname{Spec}_{\vartriangle}\operatorname{Perf} X$ for a quasi-projective variety X. As an application, we provide a new proof of the Bondal-Orlov and Ballard reconstruction theorems in terms of these spectra.

Recently, the first author introduced the Fourier-Mukai locus $\operatorname{Spec}^{\mathsf{FM}}\operatorname{Perf} X$ for a smooth projective variety X, which is constructed by gluing Fourier-Mukai partners of X inside $\operatorname{Spec}_{\vartriangle}\operatorname{Perf} X$. As another application of our main theorem, we demonstrate that $\operatorname{Spec}^{\mathsf{FM}}\operatorname{Perf} X$ can be viewed as an open ringed subspace of $\operatorname{Spec}_{\vartriangle}\operatorname{Perf} X$. As a result, we show that all the Fourier-Mukai partners of an abelian variety X can be reconstructed by topologically identifying the Fourier-Mukai locus within $\operatorname{Spec}_{\vartriangle}\operatorname{Perf} X$.

Contents

1 Introduction

We provide a new proof to the following version of the reconstruction theorem of Bondal-Orlov ([BO01]) shown by Ballard ([Bal11]).

Theorem 1.1. *Let X be a Gorenstein projective variety over an algebraically closed field with (anti-)ample canonical bundle. Then, the following assertions hold:*

2020 *Mathematics Subject Classification.* 14A10, 14F08, 14K05,18G80

Key words and phrases. abelian variety, Balmer spectrum, quasi-projective variety, perfect derived category, tensor triangulated category, triangular spectrum, triangulated category

The second author was partly supported by JSPS Grant-in-Aid for Early-Career Scientists 22K13894.

(i) *The variety X can be reconstructed solely from the triangulated category structure of the derived category* $\mathsf{Perf}\ X$ *of perfect complexes on* X.

(ii) *If there exists a Gorenstein projective variety* Y *with* $\mathsf{Perf}\ X \simeq \mathsf{Perf}\ Y$, *then* $X \cong Y$.

The basic idea of the proof is that using the Serre functor, we can reconstruct the Balmer spectrum. To provide a more detailed sketch, we recall the following constructions. In the sequel, let us assume X is a noetherian scheme unless otherwise specified.

- In [Bal02], Balmer constructed a ringed space

$$\mathrm{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X = (\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X, \mathcal{O}_{\mathsf{Perf}\ X,\otimes}),$$

  called the **Balmer spectrum**, from the tensor triangulated category $(\mathsf{Perf}\ X, \otimes^{\mathbb{L}}_X)$, where we set $\otimes^{\mathbb{L}}_X := \otimes^{\mathbb{L}}_{\mathcal{O}_X}$. This construction provides the seminal reconstruction result:

$$\mathrm{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X \cong X.$$

- In [Mat21, Mat23], one of the authors constructed a ringed space

$$\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\ X = (\mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X, \mathcal{O}_{\mathsf{Perf}\ X,\vartriangle}),$$

  called the **Matsui spectrum**[1], only using the triangulated category structure of $\mathsf{Perf}\ X$.

Note that for each spectrum, the underlying topological space consists of thick subcategories of $\mathsf{Perf}\ X$ satisfying certain conditions. For comparisons of those two spectra, there are three key results from [Mat21, Mat23, HO22], respectively:

- The Balmer spectrum is a subspace of the Matsui spectrum, topologically:

$$\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X \subset \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X;$$

- If $\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X \subset \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X$ is an open subspace, then there is an open immersion of ringed spaces

$$(\mathrm{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X)_{\mathsf{red}} \hookrightarrow \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\ X;$$

- Suppose X is a Gorenstein projective variety with (anti-)ample canonical bundle. Then, we have

$$\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X = \mathrm{Spc}^{\mathsf{Ser}} \mathsf{Perf}\ X \subset \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X$$

  where we let $\mathbb{S}$ denote the Serre functor of $\mathsf{Perf}\ X$ and define the **Serre invariant locus** to be

$$\mathrm{Spc}^{\mathsf{Ser}} \mathsf{Perf}\ X := \{\mathcal{P} \in \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X \mid \mathbb{S}(\mathcal{P}) = \mathcal{P}\} \subset \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X.$$

  Here, since each point in $\mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X$ is a certain thick subcategory of $\mathsf{Perf}\ X$, we see that the notation indeed makes sense. Note in particular that the underlying topological space of the Balmer spectrum is determined solely by the triangulated category structure of $\mathsf{Perf}\ X$ in this case.

[1] In these papers, the author introduced the ringed space under the name 'triangular spectrum', and it is called the Matsui spectrum in [HO22, HO24, Ito23].

Therefore, the following result is enough to complete the proof of Theorem 1.1.

Theorem 1.2 (Theorem 3.2)**.** *If X is a quasi-projective scheme over an algebraically closed field, then the inclusion*

$$\mathrm{Spc}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\, X \subset \mathrm{Spc}_\Delta \mathsf{Perf}\, X$$

is open. In particular, we have an isomorphism of ringed spaces:

$$(\mathrm{Spc}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\, X, \mathscr{O}_{\mathsf{Perf}\, X,\Delta}|_{\mathrm{Spc}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\, X}) \cong \mathrm{Spec}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\, X \cong X.$$

Indeed, combining the theorem with aforementioned results, we see that if X is a Gorenstein projective variety with (anti-)ample canonical bundle, then we can reconstruct X by restricting the structure sheaf of the Matsui spectrum to the Serre invariant locus, i.e.,

$$X \cong (\mathrm{Spc}^{\mathsf{Ser}} \mathsf{Perf}\, X, \mathscr{O}_{\mathsf{Perf}\, X,\Delta}|_{\mathrm{Spc}^{\mathsf{Ser}} \mathsf{Perf}\, X})$$

where the right-hand side only depends on the triangulated category structure of $\mathsf{Perf}\, X$.

Now, Theorem 1.2 has more consequences than just finishing up the new proof of the reconstruction theorem. In the rest of this paper, we will consider its implications in terms of the Fourier-Mukai locus introduced in [Ito23]. In particular, we observe that the structure sheaf on the Fourier-Mukai locus constructed in [Ito23] can be simply realized as the restriction of the structure sheaf of the Matsui spectrum (Theorem 4.5). In other words, there is an open immersion

$$\mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X \hookrightarrow \mathrm{Spec}_\Delta \mathsf{Perf}\, X$$

of ringed spaces, where $\mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X$ can be viewed as a scheme constructed by gluing copies of Fourier-Mukai partners of X realized as Balmer spectra inside the Matsui spectrum. Let us note that the Fourier-Mukai locus contains various (birational) geometric information about the Fourier-Mukai partners as observed in [Ito23] (cf. Example 4.12) and hence the categorical construction of the structure sheaf gives more paths to applications to geometry.

Finally, we give an affirmative answer to the following conjecture [Ito23, Conjecture 5.9] on the Fourier-Mukai locus of an abelian variety.

Theorem 1.3 (Theorem 5.6)**.** *Let X be an abelian variety. Then, the Fourier-Mukai locus $\mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X$ is the disjoint union of copies of Fourier-Mukai partners of X. In particular, connected components of $\mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X$ are precisely the Fourier-Mukai partners of X.*

In particular, this result together with Theorem 1.2 tells us that if we can identify the underlying topological space of the Fourier-Mukai locus inside the Matsui spectrum purely categorically, we can reconstruct all the Fourier-Mukai partners of abelian varieties as connected components of the Fourier-Mukai locus.

2 Preliminaries

Notation 2.1. Let k be a field. We assume k is algebraically closed unless otherwise specified. Throughout this paper, a triangulated category is assumed to be k-linear and **essentially small** (i.e., having a set of isomorphism classes of objects) and functors/structures are assumed to be k-linear. A variety is an integral scheme of finite type over k. For a variety, points refer to closed points. Moreover, any ring and scheme are assumed to be over k and for a scheme X, let $\mathsf{Perf}\, X$ denote the derived category of perfect complexes on X and let

$$\otimes^{\mathbb{L}}_X := \otimes^{\mathbb{L}}_{\mathcal{O}_X}$$

denote the usual derived tensor product on $\mathsf{Perf}\, X$.

Definition 2.2. Let $\mathcal{T}$ be a triangulated category.

(i) Let $\operatorname{Th}\mathcal{T}$ denote the poset of thick subcategories of $\mathcal{T}$ by inclusions. We define the **Balmer topology** on $\operatorname{Th}\mathcal{T}$ by setting open sets to be

$$U(\mathcal{E}) := \{\mathcal{I} \in \operatorname{Th}\mathcal{T} \mid \mathcal{I} \cap \mathcal{E} \neq \emptyset\}$$

for each collection $\mathcal{E}$ of objects in $\mathcal{T}$.

(ii) We say a thick subcategory $\mathcal{P}$ is a **prime thick subcategory** if the subposet

$$\{\mathcal{I} \in \operatorname{Th}\mathcal{T} \mid \mathcal{I} \supsetneq \mathcal{P}\} \subset \operatorname{Th}\mathcal{T}$$

has the smallest element. Define the **Matsui spectrum**

$$\operatorname{Spc}_{\vartriangle}\mathcal{T} \subset \operatorname{Th}\mathcal{T}$$

to be the subspace consisting of prime thick subcategories. We can equip the Matsui spectrum with a ringed space structure and let $\operatorname{Spec}_{\vartriangle}\mathcal{T} = (\operatorname{Spc}_{\vartriangle}\mathcal{T}, \mathcal{O}_{\mathcal{T},\vartriangle})$ denote the ringed space. Here, the structure sheaf $\mathcal{O}_{\mathcal{T},\vartriangle}$ is defined as the sheafification of the presheaf

$$\operatorname{Spc}_{\vartriangle}\mathcal{T} \supseteq U \mapsto Z\left(\mathcal{T}/\bigcap_{\mathcal{P}\in U}\mathcal{P}\right),$$

where

$$Z(\mathcal{S}) := \{\text{natural transformations } \eta : \mathrm{id}_{\mathcal{S}} \to \mathrm{id}_{\mathcal{S}} \text{ with } \eta[1] = [1]\eta\}$$

denotes the center of $\mathcal{S}$ for a triangulated category $\mathcal{S}$. See [Mat23] for the detailed construction.

(iii) We say a symmetric monoidal category $(\mathcal{T}, \otimes, \mathbf{1})$ is a **tensor triangulated category** (**tt-category** in short) if the bifunctor $\otimes : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ is triangulated in each variable, which is called a **tt-structure** on $\mathcal{T}$. A thick subcategory $\mathcal{I} \subset \mathcal{T}$ is said to be a **$\otimes$-ideal** if, for any $F \in \mathcal{T}$ and $G \in \mathcal{I}$, we have $F \otimes G \in \mathcal{I}$. A proper $\otimes$-ideal $\mathcal{P}$ is said to be a **prime $\otimes$-ideal** if $F \otimes G \in \mathcal{P}$ implies $F \in \mathcal{P}$ or $G \in \mathcal{P}$. Define the **Balmer spectrum**

$$\operatorname{Spc}_{\otimes}\mathcal{T} \subset \operatorname{Th}\mathcal{T}$$

to be the subspace consisting of prime $\otimes$-ideals of $(\mathcal{T}, \otimes)$. We can equip the Balmer spectrum with a ringed space structure and let $\operatorname{Spec}_{\otimes}\mathcal{T} = (\operatorname{Spc}_{\otimes}\mathcal{T}, \mathcal{O}_{\mathcal{T},\otimes})$ denote the ringed space. Here, the structure sheaf $\mathcal{O}_{\mathcal{T},\otimes}$ is defined as the sheafification of the presheaf

$$\operatorname{Spc}_{\otimes}\mathcal{T} \supseteq U \mapsto \operatorname{End}_{(\mathcal{T}/\bigcap_{\mathcal{P}\in U}\mathcal{P})}(\mathbf{1}_U)$$

of rings, where $\mathbf{1}_U$ denotes the image of $\mathbf{1}$ under the canonical functor $\mathcal{T} \to \mathcal{T}/\bigcap_{\mathcal{P}\in U}\mathcal{P}$. See [Bal05] for the detailed construction.

In the algebro-geometric setting, we have the following results:

Theorem 2.3 ([Bal05, Theorem 6.3], [Mat23, Theorem 2.12, Corollary 4.7])**.** *Let X be a noetherian scheme (over $\mathbb{Z}$). Then, we have the following assertions:*

(i) *There is a canonical isomorphism $X \cong \operatorname{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$ of ringed spaces whose underlying map is given by sending a (not necessarily closed) point $x \in X$ to a thick subcategory*

$$\mathcal{S}_X(x) := \{\mathcal{F} \in \mathsf{Perf}\, X \mid \mathcal{F}_x \cong 0 \text{ in } \mathsf{Perf}\, \mathcal{O}_{X,x}\} \subset \mathsf{Perf}\, X;$$

(ii) *A $\otimes^{\mathbb{L}}_X$-ideal of $\mathsf{Perf}\, X$ is a prime thick subcategory if and only if it is a prime $\otimes^{\mathbb{L}}_X$-ideal.*

(iii) *There is a morphism*

$$i : \operatorname{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X \to \operatorname{Spec}_{\vartriangle} \mathsf{Perf}\, X$$

of ringed spaces whose underlying continuous map is the inclusion;

(iv) *Suppose $\operatorname{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$ is open in $\operatorname{Spc}_{\vartriangle} \mathsf{Perf}\, X$. Then, the morphism*

$$i : (\operatorname{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X)_{\mathsf{red}} \to \operatorname{Spec}_{\vartriangle} \mathsf{Perf}\, X$$

is an open immersion of ringed spaces.

In [Mat23], it is shown that a quasi-affine scheme satisfies the supposition in part (iv). In the next section, we show that the supposition holds more generally if X is a quasi-projective scheme over k.

Remark 2.4. Let X be a noetherian scheme over k. We observe that Theorem 2.3 extends to this relative setting in the following sense.

(i) By definition, the structure sheaf $\mathcal{O}_{\mathsf{Perf}\, X, \otimes^{\mathbb{L}}_X}$ of the Balmer spectrum is naturally a sheaf of k-algebras and therefore the scheme $\operatorname{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$ has a canonical k-scheme structure. Now, since for a k-algebra A, the canonical ring isomorphism

$$\operatorname{End}_{\mathsf{Perf}\, A}(A) \cong \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong A$$

respects k-algebra structures, we see that the canonical isomorphism in Theorem 2.3 (i)

$$\operatorname{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X \cong X$$

is an isomorphism of k-schemes.

(ii) Similarly, the center of a k-linear triangulated category has the canonical structure of a k-algebra so that $\operatorname{Spec}_{\vartriangle} \mathsf{Perf}\, X$ is naturally a ringed space over $\operatorname{Spec} k$. Now, for a k-algebra A, the canonical evaluation ring homomorphism

$$Z(\mathsf{Perf}\, A) \to \operatorname{End}_{\mathsf{Perf}\, A}(A), \quad \eta \mapsto \eta_A$$

respects k-algebra structures. Since the open immersion

$$i : (\operatorname{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X)_{\mathsf{red}} \hookrightarrow \operatorname{Spec}_{\vartriangle} \mathsf{Perf}\, X$$

in Theorem 2.3 (iv) is essentially coming from the evaluation maps above, we see that i is an open immersion of ringed spaces over $\operatorname{Spec} k$.

3 A new proof of Bondal-Orlov reconstruction

First of all, let us see the following observation essentially made in [HO22, Proposition 5.3].

Lemma 3.1. *Let X be a quasi-projective scheme of dimension n over k and take a line bundle $\mathscr{L}$ on X. Then, the following hold:*

(i) *Take a thick subcategory $\mathcal{I} \subset \mathsf{Perf}\ X$ and an object $\mathscr{F} \in \mathcal{I}$. If $\mathscr{L}$ is very ample and there exists $d \in \mathbb{Z}$ such that*

$$\mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes d},\ \mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes (d+1)},\ \dots,\ \mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes (d+n)} \in \mathcal{I},$$

then for any $\mathscr{G} \in \mathsf{Perf}\ X$, we have

$$\mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{G} \in \mathcal{I}.$$

(ii) *Assume that $\mathscr{L}$ or $\mathscr{L}^{\otimes -1}$ is ample. A prime thick subcategory $\mathcal{P} \in \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X$ is a prime $\otimes$-ideal if and only if*

$$\mathcal{P} \otimes_X^{\mathbb{L}} \mathscr{L} = \mathcal{P}.$$

Proof. For part (i), take a thick subcategory $\mathcal{I} \subset \mathsf{Perf}\ X$ and an object $\mathscr{F} \in \mathcal{I}$ and suppose that there exists $d \in \mathbb{Z}$ such that

$$\mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes d},\ \mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes (d+1)},\ \dots,\ \mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes (d+n)} \in \mathcal{I}.$$

Then the subcategory $\mathcal{X} := \{\mathscr{G} \in \mathsf{Perf}\ X \mid \mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{G} \in \mathcal{I}\}$ is a thick subcategory of $\mathsf{Perf}\ X$ containing

$$\mathscr{L}^{\otimes d},\ \mathscr{L}^{\otimes (d+1)},\ \dots,\ \mathscr{L}^{\otimes (d+n)}.$$

By [Orl09, Theorem 4], a thick subcategory containing $\mathscr{L}^{\otimes d}, \mathscr{L}^{\otimes (d+1)}, \dots, \mathscr{L}^{\otimes (d+n)}$ needs to be $\mathsf{Perf}\ X$. Therefore, we see that $\mathcal{X} = \mathsf{Perf}\ X$; that is

$$\mathscr{F} \otimes_X^{\mathbb{L}} \mathscr{G} \in \mathcal{I}$$

for any $\mathscr{G} \in \mathsf{Perf}\ X$.

For part (ii), note that for any $\otimes$-ideal $\mathcal{P}$ in $(\mathsf{Perf}\ X, \otimes_X^{\mathbb{L}})$, we clearly have

$$\mathcal{P} \otimes_X^{\mathbb{L}} \mathscr{L} \subset \mathcal{P}$$

and the inclusion is the equality since we also have $\mathcal{P} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes -1} \subset \mathcal{P}$. If $\mathcal{P} \otimes_X^{\mathbb{L}} \mathscr{L} = \mathcal{P}$ holds, then $\mathcal{P} \otimes_X^{\mathbb{L}} \mathscr{L}^{\otimes d} = \mathcal{P}$ for any $d \in \mathbb{Z}$. Thus the converse follows from part (i) and Theorem 2.3 (ii). □

Now, we show our main theorem on the topology of the Matsui spectrum.

Theorem 3.2. *Let X be a quasi-projective scheme of dimension n over k. Then the inclusion*

$$\mathrm{Spc}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\ X \subset \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X$$

is open. In particular, we have an isomorphism of ringed spaces:

$$(\mathrm{Spc}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\ X, \mathscr{O}_{\mathsf{Perf}\ X, \vartriangle}|_{\mathrm{Spc}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\ X}) \cong (\mathrm{Spec}_{\otimes_X^{\mathbb{L}}} \mathsf{Perf}\ X)_{\mathsf{red}} \cong X_{\mathsf{red}}.$$

Proof. Take a very ample line bundle $\mathscr{L}$ on X and fix a corresponding immersion $X \hookrightarrow \mathbb{P}^N_k$ for some N. In this proof, we say an effective Cartier divisor $H \subset X$ is a **hyperplane section** if there exists a hyperplane $\tilde{H} \subset \mathbb{P}^N_k$ with $X \not\subseteq \tilde{H}$ such that $H = X \cap \tilde{H}$. For each hyperplane section $H \subset X$, define

$$\mathscr{F}_H := \mathscr{O}_H \oplus (\mathscr{O}_H \otimes^{\mathbb{L}}_X \mathscr{O}_X(H)) \oplus \cdots \oplus (\mathscr{O}_H \otimes^{\mathbb{L}}_X \mathscr{O}_X(nH))$$

and set

$$\mathcal{E}_X := \{\mathscr{F}_H \mid H \subset X \text{ is a hyperplane section}\}.$$

Now, to see $\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$ is open in $\mathrm{Spc}_{\triangle} \mathsf{Perf}\, X$, we are going to show

$$\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X = U(\mathcal{E}_X) \overset{\mathsf{def}}{=} \{\mathcal{I} \in \mathrm{Spc}_{\triangle} \mathsf{Perf}\, X \mid \mathcal{I} \cap \mathcal{E}_X \neq \emptyset\}.$$

The containment $\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X \subset U(\mathcal{E}_X)$ follows since for any (not necessarily closed) point $x \in X$, there exists a hyperplane section $x \notin H \subset X$ (as k is assumed to be algebraically closed and hence infinite) and therefore the prime $\otimes^{\mathbb{L}}_X$-ideal $\mathcal{S}_X(x)$ contains $\mathscr{F}_H$. Conversely, take a prime thick subcategory $\mathcal{P} \in U(\mathcal{E}_X)$ and take $\mathscr{F}_H \in \mathcal{P}$ for some hyperplane section $H \subset X$. Then, in particular, we have

$$\mathscr{O}_H, \mathscr{O}_H \otimes^{\mathbb{L}}_X \mathscr{O}_X(H), \ldots, \mathscr{O}_H \otimes^{\mathbb{L}}_X \mathscr{O}_X(nH) \in \mathcal{P}.$$

As the line bundle $\mathscr{O}_X(H) \cong \mathscr{L}$ is very ample, we see from Lemma 3.1 (i) that

$$\mathscr{O}_H \otimes^{\mathbb{L}}_X \mathscr{F} \in \mathcal{P}$$

holds for any $\mathscr{F} \in \mathcal{P}$ and hence

$$\mathscr{O}_X(-H) \otimes^{\mathbb{L}}_X \mathscr{F} \in \mathcal{P}$$

by considering a distinguished triangle

$$\mathscr{O}_X(-H) \otimes^{\mathbb{L}}_X \mathscr{F} \to \mathscr{O}_X \otimes^{\mathbb{L}}_X \mathscr{F} \to \mathscr{O}_H \otimes^{\mathbb{L}}_X \mathscr{F} \to \mathscr{O}_X(-H) \otimes^{\mathbb{L}}_X \mathscr{F}[1].$$

By iterating this process, we see that for any $\mathscr{F} \in \mathcal{P}$, one has

$$\mathscr{O}_X(-H) \otimes^{\mathbb{L}}_X \mathscr{F}, \mathscr{O}_X(-2H) \otimes^{\mathbb{L}}_X \mathscr{F}, \ldots, \mathscr{O}_X(-(n+1)H) \otimes^{\mathbb{L}}_X \mathscr{F} \in \mathcal{P}$$

and therefore again by Lemma 3.1 (i), we see that for any $\mathscr{G} \in \mathsf{Perf}\, X$, $\mathscr{G} \otimes^{\mathbb{L}}_X \mathscr{F} \in \mathcal{P}$. Thus, $\mathcal{P}$ is a $\otimes$-ideal and by Theorem 2.3 (ii), $\mathcal{P}$ is a prime $\otimes$-ideal, i.e., $\mathcal{P} \in \mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$. The later claim follows by Theorem 2.3 (iv). □

By Theorem 3.2, to reconstruct a reduced quasi-projective scheme over k, it is sufficient to identify the underlying topological space of corresponding Balmer spectrum in the Matsui spectrum, which is indeed done in [HO22] in the case of Gorenstein projective varieties with (anti-)ample canonical bundle. Thus, we can give a new conceptually simple proof of the following version of the Bondal-Orlov reconstruction shown by Ballard ([Bal11, Theorem 6.1]).

Theorem 3.3 (Bondal-Orlov, Ballard)**.** *Let X be a Gorenstein projective variety over k with (anti-)ample canonical bundle. Then, the following assertions hold:*

(i) *The scheme X can be reconstructed solely from the triangulated category structure of* $\mathsf{Perf}\, X$.

(ii) *If there exists a Gorenstein projective variety Y with* $\mathsf{Perf}\ X \simeq \mathsf{Perf}\ Y$*, then* $X \cong Y$.

Proof. For part (i), note from [HO22, Corollary 5.4] that we have

$$\mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X = \mathrm{Spc}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\ X \subset \mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\ X$$

where $\mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X$ denote the Serre invariant locus, i.e., the subspace of prime thick subcategories $\mathcal{P}$ satisfying $\mathbb{S}(\mathcal{P}) = \mathcal{P}$ for the Serre functor $\mathbb{S}$. (The equality is indeed a direct consequence of Lemma 3.1 (ii).) Now, by Theorem 2.3 (i), (iv) and Theorem 3.2, we have

$$X \cong \mathrm{Spec}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\ X \cong (\mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X, \mathcal{O}_{\mathsf{Perf}\ X,\vartriangle}|_{\mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X}),$$

where the right-most ringed space is determined by the triangulated category structure of $\mathsf{Perf}\ X$.

For part (ii), take a Gorenstein projective variety Y with $\Phi : \mathsf{Perf}\ X \simeq \mathsf{Perf}\ Y$. Then, we have an open embedding

$$\Phi^{-1}(\mathrm{Spc}_{\otimes^{\mathbb{L}}_Y}\,\mathsf{Perf}\ Y) \subset \mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X$$

by [Ito23, Corollary 6.3] and Theorem 3.2. Now, by Theorem 2.3 the following composition f of canonical morphisms

$$\begin{aligned} Y \cong \mathrm{Spec}_{\otimes^{\mathbb{L}}_Y}\,\mathsf{Perf}\ Y &\cong (\mathrm{Spc}_{\otimes^{\mathbb{L}}_Y}\,\mathsf{Perf}\ Y, \mathcal{O}_{\mathsf{Perf}\ Y,\vartriangle}|_{\mathrm{Spc}_{\otimes^{\mathbb{L}}_Y}\,\mathsf{Perf}\ Y}) \\ &\cong (\Phi^{-1}(\mathrm{Spc}_{\otimes^{\mathbb{L}}_Y}\,\mathsf{Perf}\ Y), \mathcal{O}_{\mathsf{Perf}\ X,\vartriangle}|_{\Phi^{-1}(\mathrm{Spc}_{\otimes^{\mathbb{L}}_Y}\,\mathsf{Perf}\ Y)}) \\ &\hookrightarrow (\mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X, \mathcal{O}_{\mathsf{Perf}\ X,\vartriangle}|_{\mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X}) \cong \mathrm{Spec}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\ X \cong X \end{aligned}$$

is an open immersion of ringed spaces. Moreover, by Remark 2.4 and by the fact that k-linear triangulated equivalence induces a k-isomorphism of the Matsui spectra, we see that f is an open immersion of k-schemes. Since both X and Y are projective and hence proper over $\mathrm{Spec}\,k$, the morphism f between them is proper (e.g. [Sta21, Tag 01W6]). In particular, f is closed and open immersion, and therefore $f : Y \to X$ is an isomorphism as X is irreducible and hence connected. □

By using similar ideas, we can get the following results as well.

Corollary 3.4. *Let X be a reduced quasi-affine scheme over k. Then, the following assertions hold:*

(i) *The scheme X can be reconstructed solely from the triangulated category structure of* $\mathsf{Perf}\ X$.

(ii) *If there exists a noetherian reduced scheme Y over k with* $\mathsf{Perf}\ X \simeq \mathsf{Perf}\ Y$*, then* $X \cong Y$.

Proof. For part (i), since X is quasi-affine (hence $\mathcal{O}_X$ is ample cf. [Sta21, Tag 0B7K]), we have an equality $\mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\ X = \mathrm{Spc}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\ X$ of sets. Indeed, the assumption shows that $\mathcal{O}_X$ is a split generator of $\mathsf{Perf}\ X$, and hence every prime thick subcategory is a prime $\otimes$-ideal. Therefore, we get an isomorphism

$$\mathrm{Spec}_{\vartriangle}\,\mathsf{Perf}\ X \cong \mathrm{Spec}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\ X \cong X$$

by Theorem 2.3.

For part (ii), take a noetherian reduced scheme Y over k with $\Phi : \mathsf{Perf}\ X \simeq \mathsf{Perf}\ Y$. Then, we have an open immersion

$$Y \hookrightarrow \mathrm{Spec}_{\vartriangle}\,\mathsf{Perf}\ Y \cong \mathrm{Spec}_{\vartriangle}\,\mathsf{Perf}\ X \cong X$$

by Theorem 3.2 and part (i). In particular, Y is also quasi-affine. Thus, part (i) shows that $X \cong \mathrm{Spec}_{\vartriangle}\,\mathsf{Perf}\ X \cong \mathrm{Spec}_{\vartriangle}\,\mathsf{Perf}\ Y \cong Y$. □

Remark 3.5. Favero [Fav12, Corollary 3.11] proved the same result under the assumptions that X is a quasi-affine variety and Y is a divisorial variety.

Remark 3.6. The arguments of the above two corollaries also prove the following more general statement:

> Let X and Y be quasi-projective schemes over k. Let $\mathscr{L}$ and $\mathscr{M}$ be line bundles on X and Y, respectively. Assume that the following two conditions:
>
> (a) $\mathscr{L}$ is (anti-)ample.
>
> (b) There is a triangulated equivalence $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ such that $\Phi(\mathscr{F} \otimes^{\mathbb{L}}_X \mathscr{L}) \cong \Phi(\mathscr{F}) \otimes^{\mathbb{L}}_Y \mathscr{M}$ for every $\mathscr{F} \in \mathsf{Perf}\, X$.
>
> Then there is an open immersion $X \hookrightarrow Y$ of k-schemes.

Further generalizations are discussed in a work in preparation ([Ito24]).

4 Categorical construction of scheme structure on Fourier-Mukai locus

In [Ito23], one of the authors studied the following locus in the Matsui spectrum.

Definition 4.1. Let $\mathcal{T}$ be a triangulated category.

(i) We say a smooth projective variety X is a **Fourier-Mukai partner** of $\mathcal{T}$ if there exists a triangulated equivalence $\mathcal{T} \simeq \mathsf{Perf}\, X$. Let $\mathrm{FM}\,\mathcal{T}$ denote the set of isomorphism classes of Fourier-Mukai partners of $\mathcal{T}$.

(ii) We say a tt-structure $\otimes$ on $\mathcal{T}$ is **geometric in** $X \in \mathrm{FM}\,\mathcal{T}$ if there exists an equivalence

$$(\mathcal{T}, \otimes) \simeq (\mathsf{Perf}\, X, \otimes^{\mathbb{L}}_X)$$

of tt-categories. We say a tt-structure $\otimes$ on $\mathcal{T}$ is **geometric** if there exists $X \in \mathrm{FM}\,\mathcal{T}$ such that $\otimes$ is geometric in X.

(iii) Define the **Fourier-Mukai locus** of $\mathcal{T}$ to be the subspace

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} := \bigcup_{\text{geom. tt-str. } \otimes \text{ on } \mathcal{T}} \mathrm{Spc}_{\otimes}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}.$$

Now, we have the following consequences of Theorem 3.2.

Proposition 4.2. *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. Then, the following hold:*

(i) *For a geometric tt-structure $\otimes$ on $\mathcal{T}$, the inclusion $\mathrm{Spc}_{\otimes}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}$ is open.*

(ii) *The Fourier-Mukai locus of $\mathcal{T}$ is open in the Matsui spectrum of $\mathcal{T}$.*

(iii) *In [Ito23], the topology on the Fourier-Mukai locus is defined to be the one generated by open subsets of the Balmer spectrum for each geometric tt-structure. This topology on $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ agrees with the subspace topology on $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ in $\mathrm{Spc}_{\vartriangle}\,\mathcal{T}$.*

Proof. For a fixed $X \in \mathrm{FM}\,\mathcal{T}$ and a tt-equivalence $\Phi : \mathcal{T} \xrightarrow{\sim} \mathsf{Perf}\ X$, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spc}_{\otimes}\,\mathcal{T} & \xrightarrow{\subseteq} & \mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} & \xrightarrow{\subseteq} & \mathrm{Spc}_{\vartriangle}\,\mathcal{T} \\ \Phi \downarrow \cong & & \Phi \downarrow \cong & & \Phi \downarrow \cong \\ \mathrm{Spc}_{\otimes}\,\mathsf{Perf}\,X & \xrightarrow{\subseteq} & \mathrm{Spc}^{\mathsf{FM}}\,\mathsf{Perf}\,X & \xrightarrow{\subseteq} & \mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\,X \end{array}$$

where the vertical maps induced from Φ are homeomorphisms. By Theorem 3.2, we have that the inclusion $\mathrm{Spc}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\ X \subset \mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\ X$ is an open embedding. Therefore, we have part (i). Part (ii) and (iii) immediately follow from part (i). □

Note by [Ito23, Theorem 4.7], we can glue the Balmer spectra corresponding to geometric tt-structures to equip $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ with a scheme structure, where the corresponding scheme is denoted by $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$. Recall by construction the scheme structure on the Fourier-Mukai locus satisfies the following properties.

Theorem 4.3 ([Ito23, Theorem 4.7])**.** *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. Then, the scheme $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ is a smooth scheme locally of finite type and for any geometric tt-structure $\otimes$, we have a canonical open immersion*

$$\mathrm{Spec}_{\otimes}\,\mathcal{T} \hookrightarrow \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$$

of schemes whose underlying continuous map is the inclusion.

Now, we show that we can obtain the same scheme structure on the Fourier-Mukai locus by simply restricting the structure sheaf on the Matsui spectrum to the Fourier-Mukai locus. To see this, let us recall the following classical result (e.g. [Gro64, Proposition 10.9.6] and [Har77, Proposition I.3.5]):

Lemma 4.4. *Let X and Y be reduced schemes locally of finite type over an algebraically closed field k and let $f, g : X \to Y$ be morphisms of schemes over k. If f and g agree on the set of closed points, then they agree as a morphism of schemes over k.*

Now, we are ready to show the following.

Theorem 4.5. *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. Then, there is an isomorphism*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} \xrightarrow{\sim} (\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}, \mathcal{O}_{\mathcal{T},\vartriangle}|_{\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}})$$

of ringed spaces whose underlying continuous map is the identity.

Proof. First, note that by Theorem 3.2, $(\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}, \mathcal{O}_{\mathcal{T},\vartriangle}|_{\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}})$ is a smooth scheme locally of finite type. Take an open covering $\{\mathrm{Spc}_{\otimes_\alpha}\,\mathcal{T}\}_{\alpha \in A}$ of $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$, where each $\otimes_\alpha$ is a geometric tt-structure on $\mathcal{T}$. Then, for each $\alpha \in A$, there exists a canonical open immersion

$$\iota_\alpha : \mathrm{Spec}_{\otimes_\alpha}\,\mathcal{T} \hookrightarrow \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$$

whose underlying continuous map is the inclusion by Theorem 4.3 and therefore there is an open immersion

$$i_\alpha : \iota_\alpha(\mathrm{Spec}_{\otimes_\alpha}\,\mathcal{T}) \hookrightarrow (\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}, \mathcal{O}_{\mathcal{T},\vartriangle}|_{\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}})$$

whose underlying continuous map is the inclusion by Theorem 3.2. Now, by Lemma 4.4, we see that $\{i_\alpha\}_{\alpha \in A}$ glues to a morphism

$$\phi : \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} \to (\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}, \mathcal{O}_{\mathcal{T},\vartriangle}|_{\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}})$$

whose underlying continuous map is the identity. Since ϕ is a homeomorphism and locally an isomorphism, it is an isomorphism. □

Remark 4.6. Note that Theorem 4.5 makes the construction of the structure sheaf on the Fourier-Mukai locus purely triangulated categorical. Therefore, if we can determine the underlying topological space of the Fourier-Mukai locus categorically, then we can reconstruct information coming from the gluings of structure sheaves of Fourier-Mukai partners performed in [Ito23]. In particular, we have more hope to have backward applications of the Fourier-Mukai locus to birational geometry.

Keeping this remark in mind, let us recall some geometric results on the Fourier-Mukai locus from [Ito23]. First, we recall basic notations and terminologies.

Definition 4.7. Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\,\mathcal{T}$.

(i) Let $\mathrm{Spec}_{\otimes,X}\,\mathcal{T} \subset \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ denote the open subscheme whose underlying topological space $\mathrm{Spc}_{\otimes,X}\,\mathcal{T}$ is the union of the Balmer spectra corresponding to geometric tt-structures in X. In other words, set

$$\mathrm{Spec}_{\otimes,X}\,\mathcal{T} := \bigcup_{\text{geom. tt-str. } \otimes \text{ in } X} \mathrm{Spec}_{\otimes}\,\mathcal{T} \subset \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}.$$

Note that for a fixed geometric tt-structure $\otimes_0$ in X, we can write

$$\mathrm{Spc}_{\otimes,X}\,\mathcal{T} = \bigcup_{\tau\in \mathrm{Auteq}\,\mathcal{T}} \tau(\mathrm{Spc}_{\otimes_0}\,\mathcal{T}),$$

where $\mathrm{Auteq}\,\mathcal{T}$ denote the group of natural isomorphism classes of triangulated autoequivalences of $\mathcal{T}$ and $\tau(\mathrm{Spc}_{\otimes_0}\,\mathcal{T})$ is the image of $\mathrm{Spc}_{\otimes_0}\,\mathcal{T}$ under the following action of $\mathrm{Auteq}\,\mathcal{T}$ on $\mathrm{Spec}_{\vartriangle}\,\mathcal{T}$:

$$\mathrm{Auteq}\,\mathcal{T} \times \mathrm{Spec}_{\vartriangle}\,\mathcal{T} \ni (\tau, \mathcal{P}) \mapsto \tau(\mathcal{P}) \in \mathrm{Spec}_{\vartriangle}\,\mathcal{T}.$$

(ii) We say X is **tt-separated** (resp. **tt-irreducible**) if $\mathrm{Spec}_{\otimes,X}\,\mathcal{T}$ is separated (resp. irreducible).

It is natural to ask how those copies of Fourier-Mukai partners interact with each other in the Matsui spectrum. Indeed, the following results and examples in [Ito23] show that the topology of the Fourier-Mukai locus is closely related to types of possible equivalences between Fourier-Mukai partners, which are then related to (birational) geometric properties of varieties.

Definition 4.8. Let X and Y be smooth projective varieties. We say a triangulated equivalence

$$\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, Y$$

is **birational** if there exists a closed point $x \in X$ such that $\Phi(k(x))$ is isomorphic to $k(y)$ for some closed point $y \in Y$, which implies that

$$\Phi(\mathrm{Spc}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\, X) \cap \mathrm{Spc}_{\otimes^{\mathbb{L}}_{\mathcal{O}_Y}}\,\mathsf{Perf}\, Y \neq \emptyset$$

and in particular that X and Y are birationally equivalent (and indeed K-equivalent) (cf. [Ito23, Lemma 4.11]). Given a birational equivalence $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, Y$, we define its **maximal domain of definition** to be the smallest open subset of X containing all closed points in $x \in X$ such that $\Phi(k(x))$ is isomorphic to $k(y)$ for some closed point $y \in Y$ (cf. [Ito23, Construction 4.6.]).

We can characterize the topology of the Fourier-Mukai locus by using birational autoequivalences as follows.

Lemma 4.9 ([Ito23, Corollary 4.21]). *Let X be a smooth projective variety. The following are equivalent:*

(i) *X is tt-separated;*

(ii) $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ *is a disjoint union of copies of X as schemes;*

(iii) *If $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, X$ is a birational triangulated equivalence, then for any closed point $x \in X$, there exists a closed point $x' \in X$ such that $\Phi(k(x)) \cong k(x')$.*

In light of Remark 4.6, a tt-separated smooth projective variety X can be reconstructed as a connected component of $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ *if we can categorically determine* $\mathrm{Spc}_{\otimes,X} \mathsf{Perf}\, X$.

Proof. Since the claims here are phrased in a little different ways from [Ito23], let us comment on how to show this version of the claims although the arguments are essentially same as the proof of [Ito23, Lemma 4.20]. First, note that part (ii) clearly implies part (i). To see part (i) implies part (iii), recall that the proof of [Ito23, Lemma 4.11 (ii)] shows for a birational autoequivalence $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, X$,

$$\Phi^{-1}(\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X) \cap \mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$$

agrees with the maximal domain of definition of Φ under the identification $X \cong \mathrm{Spec}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X$. In particular, if X is tt-separated and $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, X$ is a birational autoequivalence, then by [Ito23, Corollary 4.21], the maximal domain of definition of Φ is the whole X and hence any skyscraper sheaf gets sent to a skyscraper sheaf. Finally, part (iii) clearly implies part (ii) by [Ito23, Lemma 4.11]. □

We can say more about condition (ii) in Lemma 4.9.

Construction 4.10. Let X be a tt-separated smooth projective variety. Then the condition (iii) in Lemma 4.9 and [Huy06, Corollary 5.23] show that there is the equality of subgroups

$$\{\Phi \in \mathrm{Auteq}\, \mathsf{Perf}\, X \mid \Phi \text{ is birational up to shift}\} = \mathrm{Pic}(X) \ltimes \mathrm{Aut}(X) \times \mathbb{Z}[1] \subset \mathrm{Auteq}\, \mathsf{Perf}\, X.$$

Now, consider the set of left cosets

$$I_X := \mathrm{Auteq}\, \mathsf{Perf}\, X / \{\Phi \in \mathrm{Auteq}\, \mathsf{Perf}\, X \mid \Phi \text{ is birational up to shift}\}.$$

By [Ito23, Corollary 4.21, Theorem 4.27], we obtain an isomorphism

$$\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X \cong \bigsqcup_{I_X} X$$

of schemes.

Lemma 4.11 ([Ito23, Lemma 4.30]). *Let X be a smooth projective variety. The following are equivalent:*

(i) *X is tt-irreducible;*

(ii) *For any triangulated equivalence $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, X$, we have*

$$\Phi(\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X) \cap \mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\, X \neq \emptyset.$$

In particular, any copy of X in $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ *intersects with each other.*

(iii) *Any triangulated equivalence* $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, X$ *is birational up to shift.*

Proof. Since the wordings are a little different from [Ito23], let us comment on a proof. First of all, condition (ii) and condition (iii) are equivalent by [Ito23, Lemma 4.11 (i)]. Now, condition (i) and condition (ii) are also equivalent since by [Ito23, Theorem 4.27], $\mathrm{Spec}_{\otimes,X}\, \mathsf{Perf}\, X$ is connected if and only if any Balmer spectra corresponding to tt-structures that are geometric in X intersect with each other, where the former is equivalent to condition (i) by [Ito23, Lemma 4.30] and the latter is equivalent to condition (ii), noting that such Balmer spectra can be mapped to each other by the action of $\mathrm{Auteq}\, \mathsf{Perf}\, X$. □

Finally, let us list some examples of computations of the Fourier-Mukai locus to advertise what kind of geometry of varieties is reflected in the geometry of the Fourier-Mukai locus.

Example 4.12 ([Ito23, Example 1.1, Example 1.4])**.** Let $\mathcal{T}$ be a triangulated category with a smooth projective variety $X \in \mathrm{FM}\, \mathcal{T}$.

(i) If X is a smooth projective variety with (anti-)ample canonical bundle, then $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T} \cong X$. In particular, X is tt-irreducible and tt-separated.

(ii) If X is an elliptic curve, then $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T}$ is a disjoint union of infinitely many copies of X. In particular, X is tt-separated, but not tt-irreducible.

(iii) If X is a simple abelian variety, then all of its copies in $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T}$ are disjoint. In particular, X is tt-separated, but not tt-irreducible in general.

(iv) If X is a toric variety, then any copies of $X, Y \in \mathrm{FM}\, \mathcal{T}$ in $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T}$ intersect with each other along open sets containing tori. In particular, X is tt-irreducible and not tt-separated in general.

(v) If X is a surface containing a (-2)-curve, then the corresponding spherical twist is birational and X is not tt-separated. Moreover, X is in general not tt-irreducible either. In particular, this shows any del Pezzo surface cannot contain a (-2)-curve.

(vi) If X is connected with $X' \in \mathrm{FM}\, \mathcal{T}$ via a standard flop, then at least one pair of their copies in $\mathrm{Spec}_{\vartriangle}\, \mathcal{T}$ intersect with each other along the complement of the flopped subvarieties.

(vii) If X is a Calabi-Yau threefold, then each irreducible component of $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T}$ containing a copy of X contains all the copies of smooth projective Calabi-Yau threefolds that are birationally equivalent to X. Moreover, X is neither tt-separated nor tt-irreducible in general.

In the next section, we will generalize parts (ii) and (iii) to all abelian varieties.

5 Fourier-Mukai locus of abelian varieties

In this section, we determine the Fourier-Mukai locus associated to an abelian variety. First, let us recall some basics of the derived category of coherent sheaves on an abelian variety. For the rest of this paper, k is an algebraically closed field of characteristic 0.

Notation 5.1. Let X be an abelian variety and let $\hat{X}$ denote its dual. For a closed point $x \in X$, let $t_x : X \xrightarrow{\sim} X; y \mapsto y + x$ denote the translation. Moreover, for a closed point $\alpha \in \hat{X}$, let $\mathscr{L}_\alpha \in \mathrm{Pic}^0(X)$ denote the corresponding line bundle of degree 0.

In [Orl02], Orlov gave several important results on the derived category of coherent sheaves on an abelian variety.

Definition 5.2. Let X be an abelian variety. Then, define the group of **symplectic automorphisms** of $X \times \hat{X}$ (with respect to the natural symplectic form) to be

$$\mathrm{Sp}(X \times \hat{X}) := \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in \mathrm{Aut}(X \times \hat{X}) \middle| \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix} = \mathrm{id}_{X \times \hat{X}} \right\},$$

where $\mathrm{Aut}(X \times \hat{X})$ denotes the group of automorphisms of abelian varieties and $\hat{f}_i$ denotes the transpose of f_i. Here, we are writing an automorphism $f : X \times \hat{X} \to X \times \hat{X}$ with matrix form, where $f_1 : X \to X$, $f_2 : \hat{X} \to X$, etc. We say a symplectic automorphism f is **elementary** if f_2 is an isogeny.

Theorem 5.3 ([Orl02, Theorem 2.10, Corollary 2.13, Proposition 3.2, Construction 4.10, Proposition 4.12])**.** *Let X be an abelian variety. Then, there is a group homomorphism*

$$\gamma : \mathrm{Auteq}\,\mathsf{Perf}\, X \to \mathrm{Sp}(X \times \hat{X})$$

such that for any $\Phi_{\mathscr{E}} \in \mathrm{Auteq}\,\mathsf{Perf}\, X$ with Fourier-Mukai kernel $\mathscr{E} \in \mathsf{Perf}(X \times X)$ (which is necessarily isomorphic to a sheaf on $X \times X$ up to shift) and for any $(a, \alpha), (b, \beta) \in X \times \hat{X}$, we have that $\gamma(\Phi_{\mathscr{E}})(a, \alpha) = (b, \beta)$ if and only if

$$t_{(0,b)*}\mathscr{E} \otimes_{\mathscr{O}_{X\times X}} \pi_2^* \mathscr{L}_\beta \cong t^*_{(a,0)}\mathscr{E} \otimes_{\mathscr{O}_{X\times X}} \pi_1^* \mathscr{L}_\alpha$$

where π_i denotes projections $\pi_i : X \times X \to X$ so that $\Phi_{\mathscr{E}}(-) = \mathbb{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbb{L}}_{\mathscr{O}_{X\times X}} \mathscr{E})$. Moreover, we have

$$\mathrm{Ker}\,\gamma = (X \times \hat{X})_k \times \mathbb{Z}[1] \subset \mathrm{Auteq}\,\mathsf{Perf}\, X,$$

where each component corresponds to translations, tensor products with lines bundles of degree 0, and shifts, respectively. Furthermore, for any elementary symplectic automorphism $f \in \mathrm{Sp}(X \times \hat{X})$, there exists a (semihomogeneous) vector bundle $\mathscr{E}$ on $X \times X$ such that $\gamma(\Phi_{\mathscr{E}}) = f$.

We have the following "global" understanding of the Fourier-Mukai locus of an abelian variety:

Lemma 5.4 ([Ito23, Lemma 5.1])**.** *Let $\mathcal{T}$ be a triangulated category with an abelian variety $X \in \mathrm{FM}\,\mathcal{T}$. Then, any $Y \in \mathrm{FM}\,\mathcal{T}$ is also an abelian variety and we have*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \bigsqcup_{Y \in \mathrm{FM}\,\mathcal{T}} \mathrm{Spec}_{\otimes, Y}\,\mathcal{T}.$$

as a scheme.

In particular, in order to understand the Fourier-Mukai locus, we can focus on the locus $\mathrm{Spec}_{\otimes,X}\,\mathcal{T}$ for a single abelian variety $X \in \mathrm{FM}\,\mathcal{T}$. In [Ito23, Lemma 5.2], the following claim was only shown for an abelian variety with isomorphic dual, but it is also straightforward to show the result in general:

Proposition 5.5. *An abelian variety X is not tt-irreducible.*

Proof. By Lemma 4.11, it suffices to show there is a triangulated equivalence $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, X$ such that

$$\Phi(\mathrm{Spc}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\, X) \cap \mathrm{Spc}_{\otimes^{\mathbb{L}}_X}\,\mathsf{Perf}\, X = \emptyset.$$

Take an ample line bundle $\mathscr{L}$ on $\hat{X}$ with isogeny $\phi_{\mathscr{L}} : \hat{X} \to \hat{\hat{X}} \cong X, x \mapsto t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$. First, note that by [Plo05, Example 4.5], we have

$$\gamma(- \otimes_{\mathscr{O}_{\hat{X}}} \mathscr{L}) = \begin{pmatrix} \mathrm{id}_{\hat{X}} & 0 \\ \phi_{\mathscr{L}} & \mathrm{id}_X \end{pmatrix} \in \mathrm{Sp}(\hat{X} \times X)$$

and in particular $\phi_{\mathscr{L}} = \hat{\phi_{\mathscr{L}}}$. Thus, we have

$$f := \begin{pmatrix} \mathrm{id}_X & \phi_{\mathscr{L}} \\ 0 & \mathrm{id}_{\hat{X}} \end{pmatrix} \in \mathrm{Sp}(X \times \hat{X}).$$

Now, since f is, in particular, an elementary symplectic isomorphism, we have a vector bundle $\mathscr{E}$ on $X \times X$ such that $\Phi_{\mathscr{E}} \in \mathrm{Auteq}\,\mathsf{Perf}\ X$ (with $\gamma(\Phi_{\mathscr{E}}) = f$) by Theorem 5.3. Therefore, we see that

$$\Phi_{\mathscr{E}}(\mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X) \cap \mathrm{Spc}_{\otimes^{\mathbb{L}}_X} \mathsf{Perf}\ X = \emptyset.$$

by [Ito23, Corollary 4.10] as desired. □

Now, the following result gives an affirmative answer to [Ito23, Conjecture 5.9]:

Theorem 5.6. *An abelian variety X is tt-separated.*

Proof. Take a birational autoequivalence $\Phi \in \mathrm{Auteq}\,\mathsf{Perf}\ X$, i.e., suppose there exist $x_0, y_0 \in X$ such that $\Phi(k(x_0)) \cong k(y_0)$. By Lemma 4.9, it suffices to show that for any $x \in X$, there exists $y \in X$ such that $\Phi(k(x)) \cong k(y)$. Now, by [MGP13, Proposition 3.2], we have $\Phi = \Phi_{\mathscr{K}}$ for a sheaf $\mathscr{K}$ on $X \times X$ that is flat along each projection and therefore by [Huy06, Example 5.4] we have $\mathscr{K}|_{\{x_0\}\times X} \cong \Phi(k(x_0) \cong k(y_0)$ under the canonical identification $\{x_0\}\times X \cong X$. Moreover, by Theorem 5.3, there is a corresponding isomorphism $f_{\mathscr{K}} : X \times \hat{X} \to X \times \hat{X}$ satisfying $f_{\mathscr{K}}(a, \alpha) = (b, \beta)$ if and only if

$$t^*_{(a,0)}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_1^*\mathscr{L}_\alpha \cong t_{(0,b)_*}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_2^*\mathscr{L}_\beta,$$

which is equivalent to

$$t^*_{(a,0)}\mathscr{K} \cong t^*_{(0,-b)}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}.$$

Under canonical identifications $\{x\} \times X \cong X \cong X \times \{y\}$ for closed points $x, y \in X$, we therefore see that for any $a \in X$, we can take α, b, β such that

$$\begin{aligned}
\Phi(k(x_0 + a)) &\cong \mathscr{K}|_{\{x_0+a\}\times X} \cong (t^*_{(a,0)}\mathscr{K})|_{\{x_0\}\times X} \\
&\cong \left(t^*_{(0,-b)}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}\right)|_{\{x_0\}\times X} \\
&\cong t^*_{-b}(\mathscr{K}|_{\{x_0\}\times X}) \otimes_X \left(\pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}\right)|_{\{x_0\}\times X} \\
&\cong k(y_0 + b) \otimes_{\mathscr{O}_X} \left(\pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}\right)|_{\{x_0\}\times X} \cong k(y_0 + b)
\end{aligned}$$

as desired. □

As a combination of Construction 4.10, Lemma 5.4, and Theorem 5.6, we obtain some kind of a generalization of [Mat23, Corollary 4.10] (see also [HO22, Theorem 4.11] and [HO24, Theorem 5.3]). Here we note that $\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\ X = \mathrm{Spec}_\triangle\,\mathsf{Perf}\ X$ holds for an elliptic curve X by [HO22, Proposition 4.10] and the subsequent argument.

Corollary 5.7. *Let X be an abelian variety. Then there is an isomorphism*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\,X \cong \coprod_{Y \in \mathrm{FM}\,\mathsf{Perf}\,X} \coprod_{I_Y} Y$$

of schemes.

By Theorem 4.5 and Corollary 5.7, we can reconstruct all the Fourier-Mukai partners of an abelian variety X if we can identify the Fourier-Mukai locus

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathsf{Perf}\,X \subset \mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\,X$$

purely categorically. Along this line, the following conjecture was made in [Ito23]:

Conjecture 5.8 ([Ito23, Conjecture 6.14])**.** *Let $\mathcal{T}$ be a triangulated category with* $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. *Then, we have*

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} = \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T}.$$

The conjecture holds for curves (in particular, elliptic curves) and smooth projective varieties with (anti-) ample canonical bundle, but in [HO24], it was shown that when we have a certain K3 surface $X \in \mathrm{FM}\,\mathcal{T}$, the conjecture fails ([HO24, Theorem 5.8]). Their proof relies on the existence of spherical objects in $\mathcal{T}$ so the result does not directly generalize to abelian surfaces and we are interested if there are certain classes of abelian varieties of dimension > 1 for which the conjecture holds.

Acknowlegments

The authors thank the anonymous referee for carefully reading the paper and giving them useful comments and helpful suggestions.

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